Notes on
1.63 Advanced Environmental Fluid Mechanics
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Refs: Benoussan, Lions, & Sanchez-Palencia, *Asymptotic Analysis of Periodic Structures.* North-Holland, 1978.
Mei : *Mathematical analysis in Engineering, Cambridge* , University Press. 1997.
Lee, Sun, & Mei, Computation of permeability and dispersivities of solute or heat in periodic porous media. *Int. J. Heat & Mas Transfer*, 39(4) 661-676, 1996.

6.2 Micro-scale basis of seepage flow, Theory of homogenization

The phenomenological law of Darcy describing seepage flow in a porous medium such as soils is based on experiments. A more theoretical approach is to derive this law and predict the hydraulic conductivity. This theory is possible if the granular structure is periodic. The method of homogenization can then be applied (Sanchez-Palencia, 1974; Benoussan, et. al., 1978).

Let a rigid porous medium be saturated by an incompressible Newtonian fluid of constant density. Driven by an ambient pressure gradient, the steady flow velocity u_i and pressure p in the pores are governed by Navier-Stokes equations

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{6.2.1}$$

and

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i. \tag{6.2.2}$$

On the wetted surface Γ of the solid matrix there is no slip

$$u_i = 0 \qquad\qquad x_i \in \Gamma. \tag{6.2.3}$$

For slow flows the two terms on the right-hand side of (6.2.2), representing the pressure gradient and the viscous force, are equally important. Both the change of pressure and the flow velocity vary according to two scales: the microscale ℓ characteristic of the size of pores and grains, and the macroscale L imposed by global constraints. Again we assume that $\ell/L \ll 1$. Equating the order of magnitudes of the global pressure gradient P/L to the local viscous stress, we get

$$O\left(\frac{P}{L}\right) = O\left(\frac{\mu U}{\ell^2}\right),$$

which defines the velocity scale U. Let us normalize the space coordinates by the local scales and the unknowns according to the estimates just found

$$x_i = \ell x_i', \quad p = Pp', \quad u_i = \frac{\ell^2 P}{\mu L} u_i', \tag{6.2.4}$$

where the primed quantities are dimensionless, and p here stands for the change in pressure since it only appears in differential form. Equation (6.2.2) becomes, formally,

$$Re u_i' \frac{\partial p'}{\partial x_i'} = -\frac{1}{\epsilon} \frac{\partial p'}{\partial x_i'} + \nabla'^2 u_i', \tag{6.2.5}$$

where

$$Re \equiv \epsilon \frac{\rho \ell^2 P}{\mu^2} \tag{6.2.6}$$

is just the Reynolds number. The dimensionless continuity equation remains in the form of (6.2.1) and need not be repeated.

We assume the Reynolds number to be no greater than $O(\epsilon)$, i.e.,

$$Re \leq O(\epsilon), \quad \text{or} \quad \frac{\rho \ell^2 P}{\mu^2} \leq O(1). \tag{6.2.7}$$

Note that the pressure gradient term appears formally dominant in (6.2.5), because x_i is normalized by the micro-length scale ℓ in every term.

From here on we shall return to dimensional variables but retain the ordering symbol ϵ to keep track of the bookkeeping

$$\epsilon^2 \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \epsilon \mu \nabla^2 u_i. \tag{6.2.8}$$

The boundary condition on the wetted surface Γ of the pores is already given by (6.2.3).

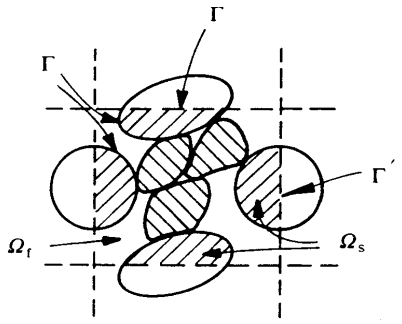

FIGURE 2. A typical Ω-cell. $\Omega = \Omega_f \cup \Omega_s$.

Figure 6.2.1: A unit cell on the microscale

Let us assume that the geometry of the porous matrix is periodic on the microscale, as depicted in Figure 6.2.1, although the structure may still change slowly over the macroscale L. Each periodic cell Ω is a rectangular box of dimension $O(\ell)$. We then expect u_i and p to be spatially periodic from cell to cell, while changing slowly over the macroscale.

We now introduce the multiple-scale coordinates

$$x_i, \qquad X_i = \epsilon x_i \tag{6.2.9}$$

and the perturbation expansions

$$u_i = u_i^{(0)} + \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \dots \tag{6.2.10}$$

$$p = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \dots, \tag{6.2.11}$$

where $u^{(j)}, p^{(j)}$ are functions of x_i and X_i.

From (6.2.1) we get at the first two orders $O(\epsilon^0)$ and $O(\epsilon)$

$$\frac{\partial u_i^{(0)}}{\partial x_i} = 0 \tag{6.2.12}$$

$$\frac{\partial u_i^{(1)}}{\partial x_i} + \frac{\partial u^{(0)}}{\partial X_i} = 0 \tag{6.2.13}$$

$$\frac{\partial u_i^{(2)}}{\partial x_i} + \frac{\partial u^{(1)}}{\partial X_i} = 0 \tag{6.2.14}$$

$$\vdots$$

Similarly we get from (6.2.8)

$$0 = -\frac{\partial p^{(0)}}{\partial x_i} \tag{6.2.15}$$

$$0 = -\frac{\partial p^{(0)}}{\partial X_i} - \frac{\partial p^{(1)}}{\partial x_i} + \mu \nabla^2 u_i^{(0)} \tag{6.2.16}$$

$$u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} = -\frac{\partial p^{(1)}}{\partial X_i} - \frac{\partial p^{(2)}}{\partial x_i} + \mu \nabla^2 u_i^{(1)} \tag{6.2.17}$$

$$\vdots$$

On the grain/water/interfaces Γ the velocity vanishes, hence

$$u_i^{(0)} = u_i^{(1)} = u_i^{(2)} = \dots = 0 \qquad x_i \in \Gamma. \tag{6.2.18}$$

In a typical Ω cell the flow must be periodic

$$u_i^{(0)}\, u_i^{(1)},\, u_i^{(2)}, \dots \quad p^{(0)}\, p^{(1)}\, p^{(1)}, \dots \qquad \text{are } \Omega \text{ periodic.} \tag{6.2.19}$$

It may be pointed out that it is the assumption (6.2.7) that renders the perturbations equations linear at each order. Had we assumed the Reynolds number to be finite, i.e., $Re = O(1)$, then the convective inertia would be of the order $O(\epsilon)$ and (6.2.16) would be replaced by

$$u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} = -\frac{\partial p^{(0)}}{\partial X_j} + \frac{\partial p^{(1)}}{\partial x_i} + \mu \nabla^2 u_i^{(0)}. \tag{6.2.20}$$

Together with (6.2.12) the cell problem is fully nonlinear (Ene and Sanchez-Palencia, 1975).

From (6.2.15) it is clear that

$$p^{(0)} = p^{(0)}(X_i). \tag{6.2.21}$$

Because of the linearity of (6.2.12) and (6.2.16) $u_i^{(0)}$ and $p^{(1)}$ can be formally represented by

$$u_i^{(0)} = -K_{ij} \frac{\partial p^{(0)}}{\partial X_j} \qquad p^{(1)} = -A_j \frac{\partial p^{(0)}}{\partial X_j} + \overline{p}^{(1)}, \tag{6.2.22}$$

where $\overline{p}^{(1)}(X_i)$ is independent of x_i. It then follows that $K_{ij}(x_i, X_i)$ and $A_j(x_i, X_i)$ must satisfy

$$\frac{\partial K_{ij}}{\partial x_i} = 0 \tag{6.2.23}$$

$$-\frac{\partial A_j}{\partial x_i} + \mu \nabla^2 K_{ij} = -\delta_{ij}, \tag{6.2.24}$$

where

$$K_{ij} = 0 \qquad \text{on } \Gamma \tag{6.2.25}$$

$$K_{ij}, \quad A_j \qquad \text{are} \quad \Omega \text{ periodic.} \tag{6.2.26}$$

These four equations define a linear boundary-value problem in an Ω cell, which must be solved numerically for any prescribed microstructure.

Defining the average over an Ω cell by

$$\langle f \rangle = \frac{1}{|\Omega|} \int_{\Omega_f} f \, d\,\Omega, \tag{6.2.27}$$

where Ω_f is the fluid volume inside the Ω cell, we get

$$\langle u_i^0 \rangle = -\langle K_{ij} \rangle \frac{\partial p^{(0)}}{\partial X_j} \tag{6.2.28}$$

$$\langle p^{(1)} \rangle = -\langle A_j \rangle \frac{\partial p^{(0)}}{\partial X_j} + n \overline{p}^{(1)}, \tag{6.2.29}$$

where n denotes the porosity, which is the ratio of fluid volume in the cell to the total cell volume

$$n = \frac{\Omega_f}{\Omega}. \tag{6.2.30}$$

The Ω average of (6.2.14) gives

$$\frac{\partial \langle u_i^{(0)} \rangle}{\partial X_i} + \frac{1}{|\Omega|} \int_{\Omega_f} \frac{\partial u_i^{(1)}}{\partial x_i} d\Omega = 0.$$

By virtue of the Gauss theorem and the boundary condition (13.6.30), the volume integral vanishes, hence

$$\frac{\partial \langle u_i^{(0)} \rangle}{\partial X_i} = 0, \tag{6.2.31}$$

which implies, in turn, that

$$\frac{\partial}{\partial X_i} \left(\langle K_{ij} \rangle \frac{\partial p^{(0)}}{\partial X_j} \right) = 0. \tag{6.2.32}$$

Equations (6.2.28) and (6.2.31) or (6.2.32) govern the seepage flow in a rigid porous medium on the macroscale(Sanchez-Palencia, 1980; Benssousan, et. al., 1978; Keller, 1980). In particular (6.2.28) is the classic Darcy's law, where $\langle K_{ij} \rangle$ is the tensor of hydraulic conductivity.

If the medium is isotropic and homogeneous on the L scale, we have

$$\langle K_{ij} \rangle = K \delta_{ij} \qquad \langle A_j \rangle = 0, \tag{6.2.33}$$

where K is a constant. It follows from (6.2.32) that

$$\frac{\partial^2 p^{(0)}}{\partial X_k \partial X_k} = 0. \tag{6.2.34}$$

With proper boundary conditions on the macroscale, $p^{(0)}$ can then be found.

Note that if the Reynolds number is of order unity, the local velocity $u^{(0)}$ and pressure $p^{(1)}$ then depend on the global pressure gradient $\partial p^{(0)}/\partial X_i$ in a nonlinear way through (13.6.32). Equations (6.2.22) and Darcy's law (6.2.28) no longer hold.

In the literature the term *homogenization* is sometimes used to mean any theoretical procedure for deriving macroscale equations by smoothing over the microscale. Homogenization by multiple scales is just one of the systematic procedures, limited to media with a periodic microstructure. In this procedure there is no need for additional closure assumptions. The derivation of effective coefficients is reduced to the solution of certain canonical boundary-value problems in a cell. One can also examine the details of the microscale once the macroscale problem is solved.

The cell problem has been solved numerically by finite elements by Sun, Lee & Mei using a Wigner-Seitz ball in each cell, as shown in Figure 6.2.2. The computed permeability is compared with the empirical formula of Carman and Kozeny in Figure 6.2.3.

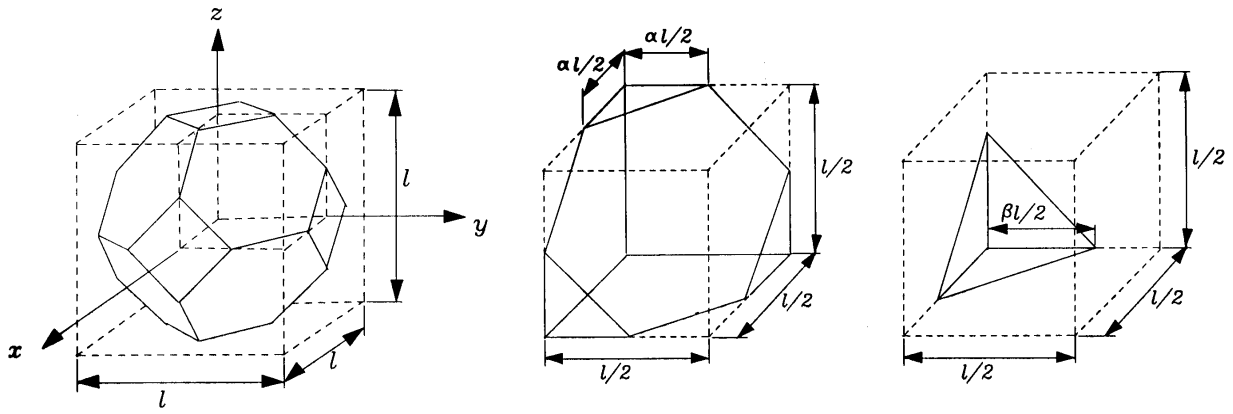

Figure 6.2.2: Left: A unit cell having one Wigner-Seitz grain. Center: 1/8th of the cell with neighboring grains in contact over a finite area. Right: 1/8th of the cell with diamond-like grains not in contact, i.e., in suspension. (From Lee, Sun & Mei. *Int'l Heat & Mass Transfer*,)

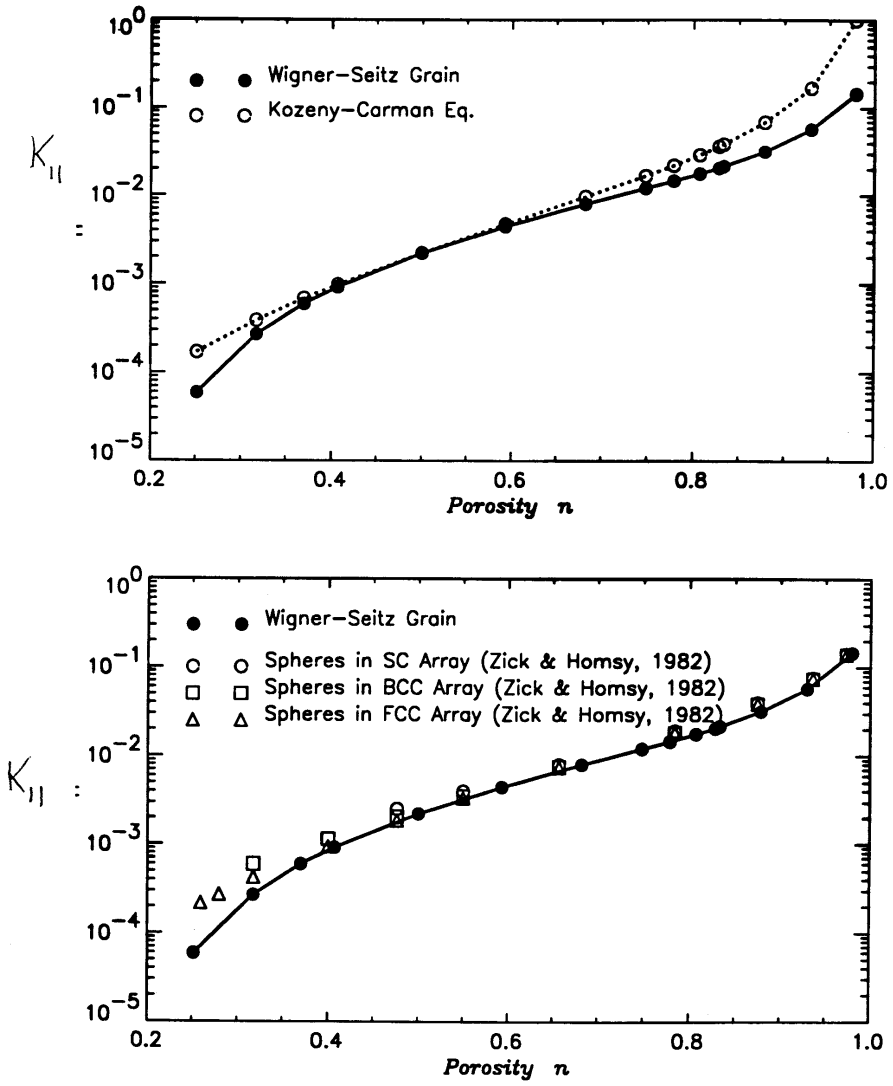

Figure 6.2.3: Comparisons of hydraulic conductivity computed for Wigner-Seitz grains by finite elements with (a) empirical formula by Carman- Kozeny (top), and with computations for uniform spheres by Zick & Homsy (bottom). (From Lee, Sun & Mei 1996)