

Notes on  
**1.63 Advanced Environmental Fluid Mechanics**  
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 2-5spreadmud.tex

## 2.5 Spreading of a shallow mass on an incline

### References:

- C. C. Mei, (1966), Nonlinear gravity waves in a thin sheet of viscous fluid, *J. Math. & Phys.* 45, 482-496.  
 K. F. Liu & C. C. Mei, 1989, Slow spreading of a sheet of a Bingham fluid on an inclined plane, *J. Fluid Mech.* 207, 505-529.  
 X. Huang & M. H. Garcia, 1997, Asymptotic solution for Bingham debris flows, *Debris -flow Hazards and Mitigation*, ASCE Proc. pp.561-575.

The spreading of a finite mass of paint, paper pulp, mud or lava on an inclined plane is of interest to a variety of industrial and geological problems. For mud modeled as a Bingham-plastic non-Newtonian fluid, Liu & Mei (1989) solved the equation similar to (2.3.22) numerically. An analytical solution for Herschel-Bulkley fluid was given later by Huang & Garcia (1997). We modify their theories for the simpler case of Newtonian fluid here.

The free surface of a thin layer is expected to flatten in time over most of the profile, but it should steepen near the downstream front where the spatial derivative is much more important than elsewhere. Let us study the problem by dividing the total fluid extent into two: the far field not too close to the steep front and the near field around the front.

### 2.5.1 Far field away from the front

Let the total length of the thin layer be  $L$  and the maximum depth be  $H$ , with  $H/L \ll 1$ . We introduce the normalized variables suitable for the far field.

$$h = Hh', \quad x = Lx', \quad t = Tt' \quad (2.5.1)$$

where  $T$  will be chosen to simplify the appearance of the final equation. Thus,

$$\frac{H}{T} \frac{\partial h'}{\partial t'} + \frac{\rho g \sin \theta}{3\mu} \frac{H^3}{L} \frac{\partial}{\partial x'} \left[ \left( 1 - \frac{H}{L \tan \theta} \frac{\partial h'}{\partial x'} \right) h'^3 \right] = 0$$

Let us choose the time scale to be

$$T = \frac{L}{\rho g H^2 \sin \theta / 3\mu} \quad (2.5.2)$$

where the denominator is the average fluid speed across the depth. The dimensionless equation for the farfield reads,

$$\frac{\partial h'}{\partial t'} + \frac{\partial}{\partial x'} \left[ \left( 1 - \frac{H}{L \tan \theta} \frac{\partial h'}{\partial x'} \right) h'^3 \right] = 0 \quad (2.5.3)$$

Assume

$$\frac{H}{L \tan \theta} \equiv \epsilon \ll 1 \quad (2.5.4)$$

(2.5.3) can then be approximated by the hyperbolic equation,

$$\frac{\partial h'}{\partial t'} + \frac{\partial h'^3}{\partial x'} = \frac{\partial h'}{\partial t'} + 3h'^2 \frac{\partial h'}{\partial x'} = 0 \quad (2.5.5)$$

It is of the class called kinematic wave equation in flood hydrology. Solution can be obtained by the theory of characteristics.

In particular (2.5.5) can be rewritten in the form

$$\frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx = dh = 0 \quad (2.5.6)$$

if

$$\frac{dx'}{dt'} = 3h'^2 \quad (2.5.7)$$

The last two equations imply that  $h'$  remains constant for all  $t'$  along the characteristic curve  $x'(t')$  defined by the differential equation (2.5.7). Moreover, if the initial profile is prescribed,

$$h'(x', 0) = h'_o(x'), \quad (2.5.8)$$

then all characteristics are straight but of different slopes. The characteristic originated from the initial point  $x' = \xi'$  at  $t' = 0$  is the straight line

$$x(t, \xi) = 3h_o^2(\xi)t + \xi, \quad (2.5.9)$$

along which

$$h'(x', t') = h'_o(\xi') \quad (2.5.10)$$

In principle we can solve for  $\xi'$  from (2.5.9) in term of  $x', t'$  and substitute the result into (2.5.10) to get  $h'(x', t')$ . For any initial hump, the characteristics at the front must intersect one another, implying multivaluedness of solution, which is physically unacceptable. The solution can still be proceeded if a discontinuity, *shock*, is allowed at  $x' = x'_s(t')$ , as long as mass is conserved.

As an example, consider a triangular initial profile:

$$h'_o(x') = \begin{cases} x' & 0 < x' < 1 \\ 0, & x' < 0, \ \& \ x' > 1. \end{cases} \quad (2.5.11)$$

The profile has a shock front to begin with, which is a mathematical idealization of course. From (2.5.9) we get

$$x' = \begin{cases} 3\xi'^2 t' + \xi', & 0 < \xi' < 1, \\ \xi', & \xi' < 0, \xi' > 1 \end{cases} \quad (2.5.12)$$

Within  $0 < \xi' < 1$ ,  $\xi'$  can be solved in terms of  $x', t'$ ,

$$\xi' = \frac{1}{6t'} \left( -1 + \sqrt{1 + 12x't'} \right) \quad (2.5.13)$$

For any  $t' > 0$ ,

$$h' = 0, \quad \text{for } x' < 0, \text{ \& } x' > x'_s(t'), \quad (2.5.14)$$

and

$$h' = \xi' = \frac{1}{6t'} \left( -1 + \sqrt{1 + 12x't'} \right), \quad 0 < x' < x'_s(t'). \quad (2.5.15)$$

The shock front  $x'_s(t')$  is unknown.

To locate the shock front  $x'_s(t')$ , we invoke mass conservation,

$$\text{initial volume} = \frac{1}{2} = \int_0^{x'_s(t')} h' dx' = \int_0^{h'_s} h' \left. \frac{dx'}{dh'} \right|_{t'} dh' \quad (2.5.16)$$

From (2.5.12),

$$x' = 3h'^2 t' + h' \quad (2.5.17)$$

so that

$$\left. \frac{dx'}{dh'} \right|_{t'} = 6h't' + 1$$

Thus

$$\frac{1}{2} = \int_0^{h'_s} h'(6h't' + 1) dh' = \frac{h_s'^2}{2} (4h_s't + 1)$$

or

$$t' = \frac{1 - h_s'^2}{4h_s'^3} \quad (2.5.18)$$

This gives the shock height  $h'_s$  implicitly as a function of  $t'$ . To get the shock position  $x'_s(t')$  we put this result into (2.5.17)

$$x'_s = \frac{3 + h_s'^2}{4h_s'} \quad (2.5.19)$$

Together (2.5.18) and (2.5.19) give  $x'_s(t')$ , as plotted in figure (2.5.1). To be needed later we note that the shock speed is given by

$$\frac{dx'_s}{dt'} = \frac{dx'_s/dt'}{dh'/dh'} = \frac{1 - 3h_s'^{-2}}{4} \frac{4h_s'^2}{1 - 3h_s'^{-2}} = h_s'^2. \quad (2.5.20)$$

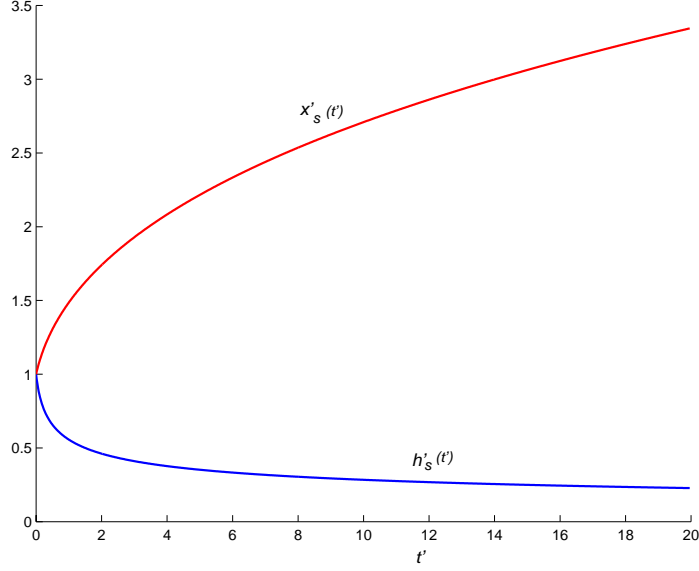


Figure 2.5.1: Time variation of the location  $x'_s(t')$  and depth  $h'_s(t')$  of shock front.

## 2.5.2 Near field of downstream front

Clearly near the shock the local free surface slope cannot satisfy (2.5.4). We define the longitudinal length scale for the near field to be  $L_s$  so that

$$\frac{H}{L_s \tan \theta} = O(1), \quad \text{i.e.,} \quad \frac{L_s}{L} = O(\epsilon) \quad (2.5.21)$$

and renormalize the near field,

$$h = HH', \quad t = Tt', \quad \frac{x - x_s(t)}{L_s} = X', \quad \text{or} \quad x' = x'_s + \epsilon X' \quad (2.5.22)$$

Thus the near field solution  $H'$  is a function of  $X'(x', t')$  and  $t'$ . The derivatives are now related by the chain rule,

$$\frac{\partial H'(x', t')}{\partial t'} = \frac{\partial H'}{\partial t'} + \frac{\partial H'}{\partial X'} \frac{\partial X'}{\partial t'} = \frac{\partial H'}{\partial t'} - \frac{1}{\epsilon} \frac{dx'_s}{dt'} \frac{\partial H'}{\partial X'} \quad (2.5.23)$$

$$\frac{\partial H'(x', t')}{\partial x'} = \frac{\partial H'}{\partial X'} \frac{\partial X'}{\partial x'} = \frac{1}{\epsilon} \frac{\partial H'}{\partial X'} \quad (2.5.24)$$

Eq. (2.5.3) becomes

$$\frac{\partial H'}{\partial t'} - \frac{1}{\epsilon} \frac{dx'_s}{dt'} \frac{\partial H'}{\partial X'} + \frac{1}{\epsilon} \frac{\partial}{\partial X'} \left[ H'^3 \left( 1 - \frac{\partial H'}{\partial X'} \right) \right] = 0 \quad (2.5.25)$$

To the leading order the near field is approximately governed by

$$-\frac{dx'_s}{dt'} \frac{\partial H'}{\partial X'} + \frac{\partial}{\partial X'} \left[ H'^3 \left( 1 - \frac{\partial H'}{\partial X'} \right) \right] = 0 \quad (2.5.26)$$

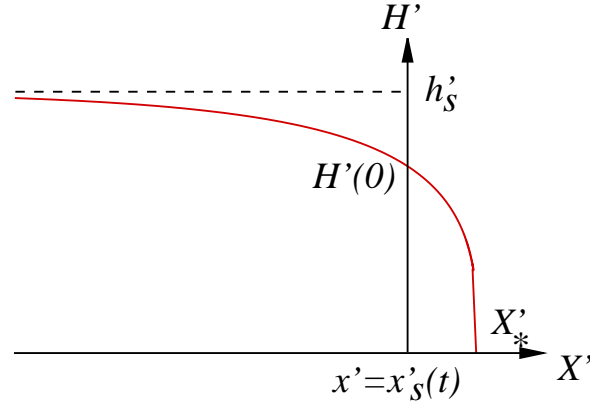


Figure 2.5.2: Matching the near-field and far-field at the front by preserving mass.

which is the ordinary differential equation for a stationary wave, moving at the known shock speed  $dx'_s/dt' = h'_s{}^2$ . Integrating with respect to  $X'$  once, we get

$$-h'_s{}^2 H' + H'^3 \left(1 - \frac{dH'}{dX'}\right) = 0.$$

The constant of integration is taken to be zero so that the surface touches the dry bed where  $H' = 0$ . We may rewrite (2.5.26) as

$$dX' = -\frac{H'^2 dH'}{h'_s{}^2 - H'^2} = dH' \left[1 - \frac{h'_s}{2} \left(\frac{1}{h'_s - H'} + \frac{1}{h'_s + H'}\right)\right]$$

It follows upon integration that

$$H' + \frac{h'_s}{2} \log \left(\frac{h'_s - H'}{h'_s + H'}\right) = X' - X'_* \quad (2.5.27)$$

This is a smooth surface decreasing monotonically from  $H' = h'_s$  at  $X' = -\infty$  to  $H' = 0$  (dry bed) at  $X' = X'_*$ . To determine the value of  $X'_*$ , we require that mass under the smooth profile be the same as that under the profile with a shock, as sketched in Figure 2.5.2.

$$\int_{-\infty}^0 (h'_s - H') dX' = \int_0^{X'_*} H'(X') dX' \quad (2.5.28)$$

Referring to Figure ??, the integrals can be equivalently carried out as

$$-\int_{H'(0)}^{h'_s} X'(H') dH' = \int_0^{H'(0)} X'(H') dH' \quad (2.5.29)$$

where  $H'(0)$  is the height at  $X' = 0$ , given by

$$H'(0) + \frac{h'_s}{2} \log \left( \frac{h'_s - H'(0)}{h'_s + H'(0)} \right) = -X'_* \quad (2.5.30)$$

Substituting (2.5.27) into (2.5.29), we get, for the left-hand-side,

$$- \int_{H'(0)}^{h'_s} X'(H') dH' = X'_* (H'(0) - h'_s) + \frac{1}{2} (H'(0)^2 - h'_s{}^2) - \frac{h'_s}{2} \int_{H'(0)}^{h'_s} \log \left( \frac{h'_s - H'}{h'_s + H'} \right) dH'$$

and for the right-hand side,

$$- \int_0^{H'(0)} X'(H') dH' = X'_* (H'(0) - 0) + \frac{1}{2} (H'(0)^2 - 0) - \frac{h'_s}{2} \int_0^{H'(0)} \log \left( \frac{h'_s - H'}{h'_s + H'} \right) dH'$$

Therefore,

$$X'_s = -\frac{1}{2} \left[ \int_0^{h'_s} \log \left( \frac{h'_s - H'}{h'_s + H'} \right) dH' + h'_s \right] = -\frac{h'_s}{2} \left[ \int_0^1 \log \left( \frac{1-z}{1+z} \right) dz + 1 \right]$$

Using the facts

$$\int_0^1 \log(1-z) dz = -1, \quad \int_0^1 \log(1+z) dz = 2 \log 2 - 1$$

we get

$$X'_* = \left( \frac{1}{2} - \log 2 \right) h'_s = 0.19897 h'_s \quad (2.5.31)$$

The near field solution is now complete.

Recall that the near and far fields coordinates are related by

$$x' = x'_s + \epsilon X' \quad (2.5.32)$$

in general and

$$x'_*(t') = x'_s(t') + \epsilon \left( \frac{1}{2} - \log 2 \right) h'_s(t') \quad (2.5.33)$$

in particular.

We can get the uniformly valid solution ( $h_{U.V.}$ ) by adding the near and far field solutions and subtracting the common part. Thus, in terms of the far-field variables,

$$h'_{U.V.} = h' + H' - h'_s \quad (2.5.34)$$

where  $h'$  is given by (2.5.15) and  $H'$  is given by (2.5.27), while the coordinates are related by (2.5.32).

Sample profiles of the free surface are shown for  $\epsilon = 0.1$  in Figure 2.5.3 for various instants.

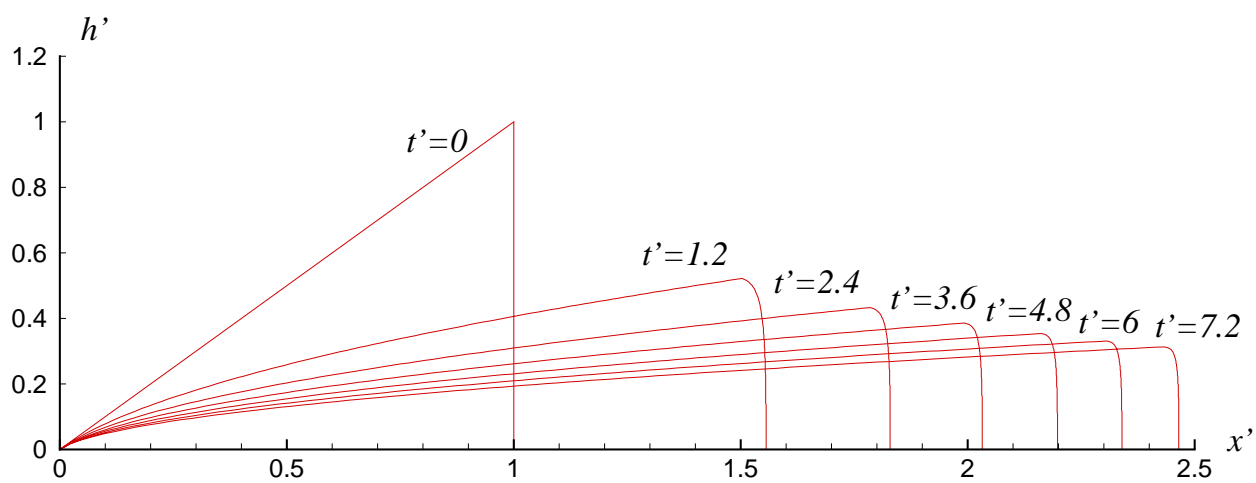


Figure 2.5.3: Slow evolution of the free surface of a viscous fluid down a dry incline. The initial profile is triangular with  $\epsilon = 0.1$ .