

Notes on
1.63 Advanced Environmental Fluid Mechanics
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 5-1KHinstab.tex

5.2 Kelvin-Helmholtz Instability for continuous shear and stratification

5.2.1 Heuristic reasoning

Due to viscosity, shear flow exists along the boundary of a jet, a wake or a plume. On the interface of salt and fresh water, density stratification further comes into play. When will dynamic instability occur?

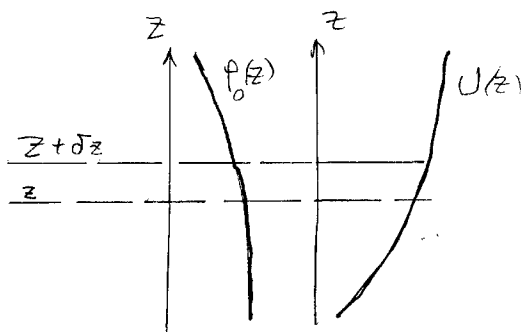


Figure 5.2.1: Exchanging fluid parcels in a stratified shear flow

Referring to Figure 5.2.1, Consider two fluid parcels, each of unit volume, at levels z and $z + dz$. Let their positions be interchanged. To overcome gravity, the force needed to lift the heavier fluid parcel by η is

$$g [\bar{\rho}(z) - \bar{\rho}(z + \eta)] = -g \frac{d\bar{\rho}}{dz} \eta.$$

Work needed to lift the heavier parcel by dz is

$$-g \frac{d\bar{\rho}}{dz} \int_z^{z+dz} \eta d\eta = -\frac{1}{2} d\bar{\rho} dz.$$

Similarly, the work needed to push the light parcel down by dz is $-\frac{1}{2} g d\bar{\rho} dz$. Therefore the total work needed is

$$-g d\bar{\rho} dz.$$

Before the exchange, the total kinetic energy is

$$\frac{1}{2}\bar{\rho}[U^2 + (U + dU)^2]$$

where Boussinesq approximation is used. After the exchange, the parcels mix with the surrounding fluid and attain the average velocity

$$(U + U + dU)/2 = U + dU/2$$

Therefore the total kinetic energy is

$$\bar{\rho}(U + dU/2)^2$$

The available kinetic energy is the difference between the kinetic energies before and after the exchange.

$$\frac{\bar{\rho}}{2} \{U^2 + (U + dU)^2 - 2(U + dU/2)^2\} = \frac{\bar{\rho}}{4}dU^2.$$

If the net available kinetic energy exceeds the work needed for the exchange, the disturbance will grow and the flow will become unstable, i.e.,

$$\frac{\bar{\rho}dU^2}{4} > -gd\bar{\rho}dz$$

Let the Richardson number be defined by

$$R_i \equiv \frac{-\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} \quad (5.2.1)$$

Instability occurs if

$$\frac{1}{4} > R_i \equiv \frac{-\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} \quad (5.2.2)$$

(Chandrasekar, 1961).

Remark: A slightly more accurate estimate can be made without Boussinesq approximation. Before the exchange, the total kinetic energy is

$$\frac{1}{2} \{ \bar{\rho}U^2 + (\bar{\rho} + d\bar{\rho})(U + dU)^2 \}.$$

After the exchange, the parcels mix with the surrounding fluid and attain the average velocity

$$(U + U + dU)/2 = U + dU/2$$

but their densities are preserved. Therefore the total kinetic energy is

$$\frac{1}{2}(\bar{\rho} + \bar{\rho} + d\bar{\rho})(U + dU/2)^2$$

The available kinetic energy is the difference between the kinetic energies before and after the exchange.

$$\frac{\bar{\rho}}{4}dU^2 - U dU d\bar{\rho} + \frac{1}{4}d\bar{\rho}dU^2$$

Ignoring the last term, the necessary condition for instability is

$$\frac{\bar{\rho}}{4}dU^2 - U dU d\bar{\rho} + \frac{1}{4}d\bar{\rho}dU^2 > -gd\bar{\rho}dz$$

or

$$\frac{1}{4} - \frac{\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz}}{\frac{1}{U}\frac{dU}{dz}} + \frac{1}{4}\frac{d\bar{\rho}}{\bar{\rho}} > \frac{-\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2}$$

On the left-hand side, the third term is negligible compared to the first. The ratio of the second term on the left to the term on the right is

$$\frac{U}{g}\frac{dU}{dz} \sim \frac{U^2}{gL}$$

where L is the length scale of stratification. As long as the last ratio is very small, the criterion $R_i < 1/4$ still holds.

Let us confirm the heuristic result but the linearize theory.

5.2.2 Linearized instability theory for continuous shear and stratification.

Let the total flow field be $(U+u, w, P+p, \bar{\rho}+\tilde{\rho})$ where $U, P, \bar{\rho}$ represent the background flow $(u, w, p, \tilde{\rho})$ the dynamical perturbations of infinitesimal magnitude. The linearized governing equations are: continuity:

$$u_x + w_z = 0 \tag{5.2.3}$$

incompressibility:

$$\tilde{\rho}_t + U\tilde{\rho}_x + w\bar{\rho}' = 0 \tag{5.2.4}$$

where

$$\bar{\rho}' \equiv \frac{d\bar{\rho}}{dz}$$

and momentum conservation:

$$\bar{\rho}(u_t + Uu_x + wU_z) = -p_x \tag{5.2.5}$$

$$\bar{\rho}(w_t + Uw_x) = -p_z - \tilde{\rho}g. \tag{5.2.6}$$

where $\tilde{\rho}$ denotes the perturbation of density from $\bar{\rho}$.

Let us follow Miles and introduce a new unknown η by enoting $\tilde{\rho} = -\bar{\rho}'\eta$, then Eqn. (5.2.4) gives

$$\eta_t + U\eta_x = w \tag{5.2.7}$$

Consider

$$\eta = F(z)e^{ik(x-ct)}, \quad (5.2.8)$$

where

$$c = \omega/k = c_r + ic_i.$$

For fixed k the flow is unstable if $c_i > 0$, since

$$e^{-ikct} = e^{-ikc_r t} e^{kc_i t}.$$

Let

$$\{u, w, p, \tilde{\rho}\} = \{\hat{u}(z), \hat{w}(z), \hat{p}(z), -\tilde{\rho}'F(z)\} e^{ik(x-ct)} \quad (5.2.9)$$

We get from Eqn. (5.2.7)

$$\hat{w} = ik(U - c)F, \quad (5.2.10)$$

from Eqn. (5.2.3)

$$\hat{u} = -[(U - c)F]', \quad (5.2.11)$$

and from Eqn. (5.2.5)

$$\tilde{\rho}(ik(U - c)\hat{u} + U'[ik(U - c)F]) = \hat{p}ik$$

or

$$\tilde{\rho}[(U - c)(-)[(U - c)F]' + U'(U - c)F] = \hat{p},$$

hence

$$\hat{p} = \tilde{\rho}(U - c)^2 F'. \quad (5.2.12)$$

Substituting Eqns. (5.2.9), (5.2.10), (5.2.11) and (5.2.12) into Eqn. (5.2.6), we get

$$[\tilde{\rho}(U - c)^2 F']' + \tilde{\rho}[N^2 - k^2(U - c)^2]F = 0, \quad (5.2.13)$$

where N is the Brunt-Väisälä frequency defined by

$$N^2 = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}. \quad (5.2.14)$$

Let the top and bottom be rigid walls, then $w = 0$. Hence,

$$\eta = 0 \quad \text{i.e.,} \quad F = 0, \quad z = 0, d. \quad (5.2.15)$$

The argument is unchanged if the top and bottom are at $z = \infty$ and $z = -\infty$. Equations (5.2.13) and (5.2.15) constitute an eigenvalue problem where $c = c_r + ic_i$ is the eigenvalue. If $c_i > 0$, instability occurs.

5.2.3 A necessary condition for instability (J.W. Miles, L. N. Howard).

For brevity we set $W = U - c$. Miles further introduce $G = \sqrt{W}F$, so that Eqn. (5.2.13) becomes

$$(\bar{\rho}WG')' - \left[\frac{1}{2}(\bar{\rho}U')' + k^2\bar{\rho}W + \frac{\bar{\rho}}{W} \left(\frac{1}{4}U'^2 - N^2 \right) \right] G = 0. \quad (5.2.16)$$

The boundary conditions are

$$G(0) = G(d) = 0. \quad (5.2.17)$$

Multiplying Eqn. (5.2.16) by G^* and integrating by parts

$$\int_0^d \left\{ \bar{\rho}W (|G_1'|^2 + k^2 |G_1|^2) + \frac{1}{2}(\bar{\rho}U')' |G|^2 + \bar{\rho} \left(\frac{1}{4}U'^2 - N^2 \right) W^* \left| \frac{G}{W} \right|^2 \right\} dz = 0. \quad (5.2.18)$$

We now seek the necessary condition for instability, i.e., $c_i \neq 0$. Writing

$$W = (U - c_r) - ic_i \quad W^* = (U - c_r) + ic_i$$

and substituting these in (5.2.18), we get

$$\int_0^d \left\{ \bar{\rho}(U - c_r - ic_i) (|G'|^2 + k^2 |G|^2) + \frac{1}{2}(\bar{\rho}U')' |G|^2 + \bar{\rho} \left(\frac{1}{4}U'^2 - N^2 \right) (U - c_r + ic_i) \left| \frac{G}{W} \right|^2 \right\} dz = 0.$$

Separating the imaginary part, we get, if $c_i \neq 0$,

$$\int_0^d \bar{\rho} \left((|G'|^2 + k^2 |G|^2) dz + \int_0^d \bar{\rho} \left(g\beta - \frac{1}{4}(U')^2 \right) \left| \frac{G}{W} \right|^2 dz \right) = 0.$$

For this to be true it is necessary that $N^2 < \frac{1}{4}(U')^2$ or

$$R_i = \frac{N^2}{(U')^2} = \frac{-\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}}{\left(\frac{dU}{dz} \right)^2} < \frac{1}{4}. \quad (5.2.19)$$

This confirms the heuristic result as the necessary (but not sufficient) condition for instability (J.W. Miles, L. N. Howard).