October 30, 2002

Chapter 5. Rudiments of hydrodynamic instability

References:
Drazin: **Introduction to hydrodynamic stability**
Chandrasekar: **Hydrodynamic and hydromagnatic instability**
Stuart: **Hydrodynamic stability**, in Rosenhead (ed) : **Laminar boundary layers**
Lin : **Theory of Hydrodynamic stability**
Drazin and Reid : **Hydrodynamic stability**
Schlichting: **Boundary layer theory**
C.S. Yih, 1965, **Dynamics of Inhomogeneous Fluids**, MacMillan.

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Various factors can be crucial to hydrodynamic instability.: shear, gravity, surface tension, heat, centrifugal force, etc. Some instabilities lead to a different flow; some to turbulence.
    We only discuss the linearized analysis of instability.

    Let us take the parallel flow as an example and outline the standard procedure of linearized analysis as follows:

1. Let the basic flow have velocity $U(z)$ in the x direction.

2. Derive the linearized equation for infinitestimal perturbations and homogeneous boundary conditions.

3. Consider sinusoidal wave- like disturbances so that all unknowns are of the form
$$f_i = \bar{f}_i(z)e^{i(kx-\omega t)}$$

4. Reduce the partial differential equations to ordinary differential equation(s), hence obtain an eignevalue problem.

5. Solve for the eigenvalue ω for a given and real k.

6. If the eigenvalue is complex, $\omega = \omega_r + i\omega_i$, find the condition under which $\omega_i > 0$. Since
$$e^{-i\omega t} = e^{-i(\omega_r + i\omega_i)t} = e^{-i\omega_r t}e^{\omega_i t},$$
the flow is unstable $\omega_i > 0$ as time grows, stable if $\omega_i < 0$, and neutrally stable if $\omega_i = 0$.

5.1 Kelvin-Helmholtz instability of flow with discontinuous shear and stratification.

5.1.1 Nonlinear formulation

Consider two immiscible fluids. The upper ($z > 0$) and lower ($z < 0$) fluids have different densities ρ_1 and ρ_2. Before any perturbations the lower fluid is stationary, while the upper fluid moves at the steady and uniform velocity $\bar{q}_o = U\vec{i}$. With gravity, this is called the Kelvin-Helmholtz instability problem. For the time being, let us assume that there is no gravity.

Now let there be a disturbance on the interface whose vertical displacement is $z = \zeta(x,t)$. The corresponding disturbances of velocity and pressure are $\vec{q}_1$ and p_1 in the upper fluid and $\vec{q}_2$ and p_2 in the lower fluid.

Let us first formulate the nonlinear problem. In the upper fluid, we have

$$\vec{q}_1 = \nabla\Phi_1, \quad \text{and} \quad \nabla^2\Phi_1 = 0, \quad z > 0 \tag{5.1.1}$$

and the Bernoulli equation,

$$\frac{\partial\Phi_1}{\partial t} + \frac{(\nabla^2\Phi_1)^2}{2} = -\frac{p_1}{\rho_1} + C_1 \tag{5.1.2}$$

In the lower fluid, we have :

$$\vec{q}_2 = \nabla\Phi_2, \quad \text{and} \quad \nabla^2\Phi_2 = 0 \quad z < 0 \tag{5.1.3}$$

with the Bernoulli equation,

$$\frac{\partial\Phi_2}{\partial t} + \frac{(\nabla^2\Phi_2)^2}{2} = -\frac{p_2}{\rho_2} + C_2 \tag{5.1.4}$$

On the interface $F = z - \zeta(x,t) = 0$ the kinematic conditions are

$$\frac{\partial F}{\partial t} + \nabla\Phi_1 \cdot \nabla F = 0 \tag{5.1.5}$$

and

$$\frac{\partial F}{\partial t} + \nabla\Phi_2 \cdot \nabla F = 0 \tag{5.1.6}$$

or, more explicitly,

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\Phi_1}{\partial x}\frac{\partial\zeta}{\partial x} = \frac{\partial\Phi_1}{\partial z}, \quad z = \zeta, \tag{5.1.7}$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\Phi_2}{\partial x}\frac{\partial\zeta}{\partial x} = \frac{\partial\Phi_2}{\partial z}, \quad z = \zeta, \tag{5.1.8}$$

The dynamic condition is

$$p_1 = p_2, \quad \text{or} \quad z = \zeta. \tag{5.1.9}$$

$$\rho_1 \left(\frac{\partial \Phi_1}{\partial t} + \frac{(\nabla^2 \Phi_1)^2}{2} - C_1 \right) = \rho_2 \left(\frac{\partial \Phi_2}{\partial t} + \frac{(\nabla^2 \Phi_2)^2}{2} - C_2 \right), \quad \text{or} \quad z = \zeta. \tag{5.1.10}$$

Far away from the interface,

$$\nabla \Phi_1 \to U_1 \vec{i}, \quad z \to \infty, \tag{5.1.11}$$

$$\nabla \Phi_2 \to U_2 \vec{i}, \quad z \to -\infty. \tag{5.1.12}$$

Note that in the absence of any transient disturbances, $\nabla \Phi_1 = U_1 \vec{i}, \nabla \Phi_2 = U_2 \vec{i}$, so that

$$\rho_1 \left(C_1 - \frac{U_1^2}{2} \right) = \rho_2 \left(C_2 - \frac{U_2^2}{2} \right) \tag{5.1.13}$$

5.1.2 Linearization

We now limit our selves to infinitesimal disturbances and assume that the interface displacement is infinitesimal when compared to other length scales.

$$\Phi_1 = U_1 x + \phi_1, \quad and \quad \Phi_2 = U_2 x + \phi_2, \tag{5.1.14}$$

with $\phi_n \ll \Phi_n, n = 1, 2$ and linearize all the conditions. We then get

$$\nabla^2 \Phi_1 = 0, \quad z > 0, \tag{5.1.15}$$

nad

$$\nabla^2 \Phi_2 = 0, \quad z < 0, \tag{5.1.16}$$

Note that

$$f(x, \zeta, t) = f(x, 0, t) + \zeta \frac{\partial f}{\partial z}|_{z=0} + \cdots, \tag{5.1.17}$$

On the inteface we have ther kinematic conditions

$$\frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_1}{\partial z}, \quad z = 0, \tag{5.1.18}$$

$$\frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_2}{\partial z}, \quad z = 0, \tag{5.1.19}$$

and the dynamic condition

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right), \quad z = 0. \tag{5.1.20}$$

where (5.1.13) has been used.

5.1.3 Normal mode analysis

Assume a sinusoidal disturbance

$$\zeta = Ae^{i(kx-\omega t)} \tag{5.1.21}$$

We follow the usual convention that the real part of the final solution is to be taken to represent physical quaitities. To satisfy Laplace equation we take

$$\phi_1 = \bar{\phi}_1 \, e^{-kz} e^{i(kx-\omega t)} \tag{5.1.22}$$

$$\phi_2 = \bar{\phi}_2 \, e^{kz} e^{i(kx-\omega t)} \tag{5.1.23}$$

Here $A, \bar{\phi}_1, \bar{\phi}_2$ are unknown constants.

Substituting into boundary conditions on the interface eqs (5.1.18-5.1.20) we get three homogeneous alegebric equations for $A, \bar{\phi}_1, \bar{\phi}_2$,

$$(-i\omega + ikU_1)A = -k\bar{\phi}_1 \tag{5.1.24}$$

$$(-i\omega + ikU_2)A = k\bar{\phi}_2 \tag{5.1.25}$$

$$\rho_1(-i\omega + ikU_1)\phi_1 = \rho_2(-i\omega + ikU_2)\phi_2$$

which can be written in matrix form

$$\begin{bmatrix} -i\omega + ikU_1 & k & 0 \\ -i\omega + ikU_2 & 0 & -k \\ 0 & \rho_1(-i\omega + ikU_1) & \rho_2(-i\omega + ikU_2) \end{bmatrix} \left\{ \begin{array}{c} A \\ \bar{\phi}_1 \\ \bar{\phi}_2 \end{array} \right\} = 0$$

In order to get non trivial solutions the coefficient determinant of the algebraic equations must vanish, leading to the eigenvalue condition

$$\rho_1(-\omega + kU_1)^2 + \rho_2(-\omega + kU_2)^2 = 0$$

or

$$(\rho_1 + \rho_2)\omega^2 - 2k(\rho_1 U_1 + \rho_2 U_2)\omega + k^2(\rho_1 U_1^2 + U\rho_2 U_2^2) = 0$$

The solution is

$$\omega = k\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm ik\frac{\sqrt{\rho_1 \rho_2}|U_1 - U_2|}{\rho_1 + \rho_2} \tag{5.1.26}$$

The lower sign corresponds to instability. Thus a velocity discontinuity is always unstable. This is the limiting case of the continuous velocity profile with a point of inflection.

5.1.4 Physical interpretation

Let us use the mathematical results to see the physics (Batchelor, p 516ff). For simlicity we choose $\rho_1 = \rho_2 = \rho$ and $U_1 = -U_2 = U > 0$ so that (5.1.26) reduces to

$$\omega = \pm ikU \tag{5.1.27}$$

and is purely imaginary. The positive root (upper sign) signifies instability.

The basic flow has a vortex sheet at $z = 0$. Due to the disturbance, the vorticity distribution is changed. Across the interface the total vorticity is approximately

$$\int_{0_-}^{0_+} \frac{\partial u'}{\partial z} dz = u_1'(z = 0_+) - u_2'(z = 0_-) = \frac{\partial \phi_1}{\partial x}\big|_0 - \frac{\partial \phi_2}{\partial x}\big|_0$$

Note that a positive vorticity vector points into the paper (as does the y axis); the circulation is clockwise in the $x - z$ plane. From the solution (5.1.24) and (5.1.25) we get

$$\frac{\partial \phi_1}{\partial x}\big|_0 = ik\bar\phi_1 e^{ikx-i\omega t} = -i(-i\omega + ikU_1)Ae^{ikx-i\omega t} = -i(-i\omega + ikU_1)\zeta$$

and

$$\frac{\partial \phi_2}{\partial x}\big|_0 = ik\bar\phi_2 e^{ikx-i\omega t} = i(-i\omega + ikU_2)Ae^{ikx-i\omega t} = i(-i\omega + ikU_2)\zeta$$

Therefore the vorticity along the interface is

$$[-2\omega + k(U_1 + U_2)]\zeta = \mp 2ikU\zeta = 2kUA\exp(ikx \mp i\pi/2) \pm kUt)$$

Consider only the upper signs for the unstable solution,

$$-2ikU\zeta = 2kUA\exp\left[ik\left(x - \frac{\pi}{2k}\right) + kUt\right] = 2kUA\exp\left[ik\left(x - \frac{\lambda}{4}\right) + kUt\right]$$

since $k = 2\pi/\lambda$ where λ is th wave length. In real form the vorticity disturbance varies as

$$2kUA\cos\left[k\left(x - \frac{\lambda}{4}\right)\right]e^{kUt}$$

whereas the interface varies as

$$A\cos kx\, e^{kUt}$$

The vorticity disturbance varies along the interface and leads the interface displacement ζ in phase by $\pi/2$ or one quarter of a wavelength. As sketched in the upper part of Figure 5.1.1, the vorticity disturbance is at the positive (clockwise) maximum at points like A where $\zeta = 0$ and $\zeta_x < 0$ and at the negative (counterclockwise) maximum at points like B where $\zeta = 0$ and $\zeta_x > 0$, being zero at the crests and troughs of the interface. All these vortices tend to lift the interface crests and supress the troughs.
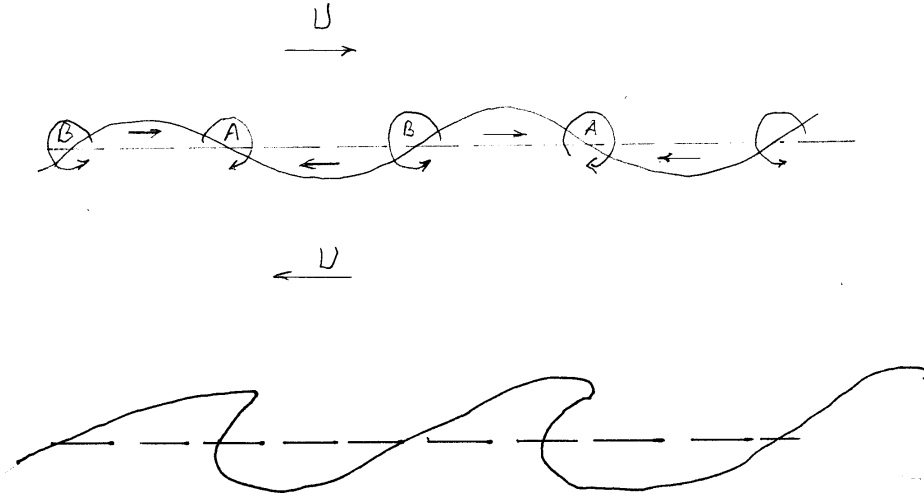
Figure 5.1.1: Disturbance vorticity along a velocity dicontinuity. $U_1 = U > 0, U_2 = -U < 0$.

Moreover the disturbances are convected by the average velocity

$$\frac{\partial \phi_1}{\partial x}\big|_0 + \frac{\partial \phi_2}{\partial x}\big|_0 = k(U_1 - U_2)\zeta = 2U\zeta$$

which is positive near the crests and negative near the troughs. Therefore clockwise vorticity disturbances accumulate around A, while the counterclockwise disturbances are swept away from and thinned out around B. Because of this assymetry the wavy interface not only tend to amplify but to curl up in the forward direction as sketched in the lower part of Figure 5.1.1.

After some time the disturbance grows so large that nonlinearity must be accounted for. Numerical nonlinear analysis as well as laboratory experiments show that the interface curls up, as can be seen in the famous photograph by Thorpe: Figure5.1.2.

Homework K-H instability: Show that when gravity and surface tension are both present, the eigenvalue condition is

$$\frac{\omega}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[-\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} + \frac{Tk}{\rho_1 + \rho_2} \right]^{1/2} \tag{5.1.28}$$

Discuss the condition for instability.

Remark: The special case of $U_1 = U_2 = 0$ but $g \neq 0$, i.e., the instability of normally accelerated density discontinuity, is called the Rayleigh-Taylor instability. It is of interest when an interface is acelerated in the normal direction, and has a wide range of industrial and astrophysical applcations.

Remark: The K-H instabilty problem can be generalized in a number of ways, by including viscosity and continuous density and/or velocity profiles, see Chandrasekhar.

145. Kelvin-Helmholtz instability of stratified shear flow. A long rectangular tube, initially horizontal, is filled with water above colored brine. The fluids are allowed to diffuse for about an hour, and the tube then quickly tilted six degrees, setting the fluids into motion. The brine accelerates uniformly down the slope, while the water above similarly accelerates up the slope. Sinusoidal instability of the interface occurs after a few seconds, and has here grown nonlinearly into regular spiral rolls. *Thorpe 1971*
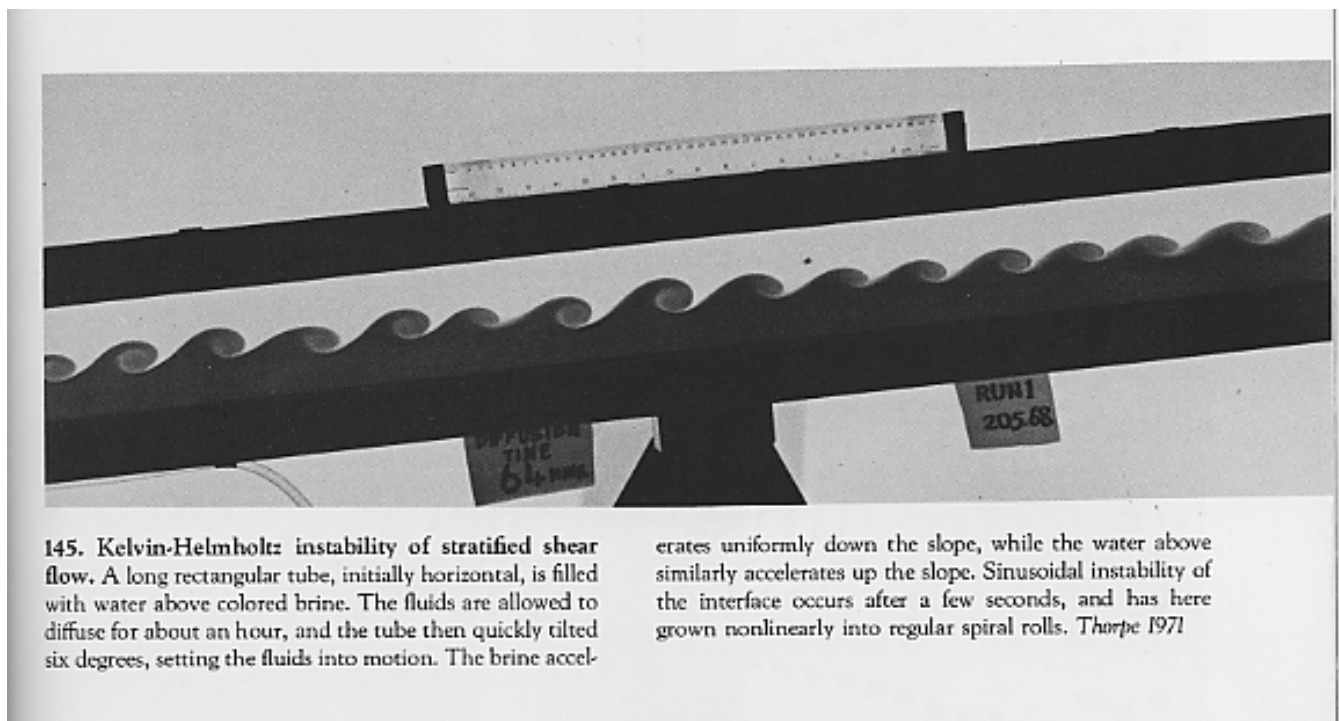
Figure 5.1.2: Kelvin-Helmholtz instability in the laboratory.