**I-campus project**

**School-wide Program on Fluid Mechanics**

**Modules on High Reynolds Number Flows**

K. P. Burr, T. R. Akylas & C. C. Mei

**CHAPTER TWO**

**TWO-DIMENSIONAL LAMINAR BOUNDARY LAYERS**

# 1   Introduction.

When a viscous fluid flows along a fixed impermeable wall, or past the rigid surface of an immersed body, an essential condition is that the velocity at any point on the wall or other fixed surface is zero. The extent to which this condition modifies the general character of the flow depends upon the value of the viscosity. If the body is of streamlined shape and if the viscosity is small without being negligible, the modifying effect appears to be confined within narrow regions adjacent to the solid surfaces; these are called *boundary layers*. Within such layers the fluid velocity changes rapidly from zero to its main-stream value, and this may imply a steep gradient of shearing stress; as a consequence, not all the viscous terms in the equation of motion will be negligible, even though the viscosity, which they contain as a factor, is itself very small.

A more precise criterion for the existence of a well-defined laminar boundary layer is that the Reynolds number should be large, though not so large as to imply a breakdown of the laminar flow.

# 2   Boundary Layer Governing Equations.

In developing a mathematical theory of boundary layers, the first step is to show the existence, as the Reynolds number $R$ tends to infinity, or the kinematic viscosity $\nu$ tends to zero, of a limiting form of the equations of motion, different from that obtained by putting $\nu = 0$ in the first place. A solution of these limiting equations may then reasonably be expected to describe approximately the flow in a laminar boundary layer

for which $R$ is large but not infinite. This is the basis of the classical theory of laminar boundary layers.

The full equation of motion for for a two-dimensional flow are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.3}$$

where the $x$ and $y$ variables are, respectively, the horizontal and vertical coordinates, $u$ and $v$ are, respectively, the horizontal and vertical fluid velocities and $p$ is the fluid pressure. A wall is located in the plane $y = 0$. We consider non-dimensional variables

$$x' = \frac{x}{L}, \tag{2.4}$$

$$y' = \frac{y}{\delta}, \tag{2.5}$$

$$u' = \frac{u}{U}, \tag{2.6}$$

$$v' = \frac{v}{U}\frac{L}{\delta}, \tag{2.7}$$

$$p' = \frac{p}{\rho U^2}, \tag{2.8}$$

$$t' = t\frac{U}{L}, \tag{2.9}$$

where $L$ is the horizontal length scale, $\delta$ is the boundary layer thickness at $x = L$, which is unknown. We will obtain an estimate for it in terms of the Reynolds number $R$. $U$ is the flow velocity, which is aligned in the $x$-direction parallel to the solid boundary. The non-dimensional form of the governing equations is:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\nu}{UL}\frac{\partial^2 u'}{\partial (x')^2} + \frac{\nu}{UL}\frac{L^2}{\delta^2}\frac{\partial^2 u'}{\partial (y')^2}, \tag{2.10}$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -\left(\frac{L}{\delta}\right)^2\frac{\partial p'}{\partial y'} + \frac{\nu}{UL}\frac{\partial^2 v'}{\partial (x')^2} + \frac{\nu}{UL}\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v'}{\partial (y')^2}, \tag{2.11}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \tag{2.12}$$

where the Reynolds number for this problem is

$$R = \frac{UL}{\nu}. \tag{2.13}$$

Inside the boundary layer, viscous forces balance inertia and pressure gradient forces. In other words, inertia and viscous forces are of the same order, so

$$\frac{\nu}{UL}\left(\frac{L}{\delta}\right)^2 = O(1) \Rightarrow \delta = O(R^{-1/2}L). \tag{2.14}$$

Now we drop the primes from the non-dimensional governing equations and with equation (2.14) we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{2.15}$$

$$\frac{1}{R}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{1}{R^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2.16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.17}$$

In the limit $R \to \infty$, the equations above reduce to:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{2.18}$$

$$-\frac{\partial p}{\partial y} = 0, \tag{2.19}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.20}$$

Notice that according to equation (2.19), the pressure is constant across the boundary layer. In terms of dimensional variables, the system of equations above assume the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.21}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \tag{2.22}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.23}$$

To solve the system of equations above we need to specify initial and boundary conditions.

# 3    Steady State Laminar Boundary Layer on a Flat Plate.

We consider a flat plate at $y = 0$ with a stream with constant speed $U$ parallel to the plate. We are interested in the steady state solution. We are not interested in how the flow outside the boundary layer reached the speed $U$. In this case, we need only to consider boundary conditions and the equation (2.18) simplifies since $\frac{\partial u}{\partial t} = 0$. At the plate surface there is no flow across it, which implies that

$$v = 0 \text{ at } y = 0. \tag{3.24}$$

Due to the viscosity we have the no slip condition at the plate. In other words,

$$u = 0 \text{ at } y = 0. \tag{3.25}$$

At infinity (outside the boundary layer), away from the plate, we have that

$$u \to U \text{ as } y \to \infty. \tag{3.26}$$

For the flow along a flat plate parallel to the stream velocity $U$, we assume no pressure gradient, so the momentum equation in the $x$ direction for steady motion in the boundary layer is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{3.27}$$

and the appropriate boundary conditions are:

$$\text{At } y = 0, x > 0; u = v = 0, \tag{3.28}$$

$$\text{At } y \to \infty, \text{all } x; u = U, \tag{3.29}$$

$$\text{At } x = 0; u = U. \tag{3.30}$$

These conditions demand an infinite gradient in speed at the leading edge $x = y = 0$, which implies a singularity in the mathematical solution there. However, the assumptions implicit in the boundary layer approximation break down for the region of slow flow around the leading edge. The solution given by the boundary layer approximation is not valid at the leading edge. It is valid downstream of the point $x = 0$. We would like to reduce the boundary layer equation (3.27) to an equation with a single dependent variable. We consider the stream function $\psi$ related to the velocities $u$ and $v$ according to the equations

$$u = \frac{\partial \psi}{\partial y}, \tag{3.31}$$

$$v = -\frac{\partial \psi}{\partial x}. \tag{3.32}$$

If we substitute equations (3.31) and (3.32) into the equation (3.27), we obtain a partial differential equation for $\psi$, which is given by

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}, \tag{3.33}$$

and the boundary conditions (3.28) and (3.29) assume the form

$$\text{At } y = 0, x > 0; \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} = 0, \tag{3.34}$$

$$\text{At } y \to \infty, \text{ all } x; \frac{\partial \psi}{\partial y} = U. \tag{3.35}$$

The boundary value problem admits a similarity solution. We would like to reduce the partial differential equation (3.33) to an ordinary differential equation. We would like to find a change of variables which allows us to perform the reduction mentioned above.

## 3.1   Similarity Solution.

We look for a one-parameter transformation of variables $y, x$ and $\psi$ under which the equations for the boundary value problem for $\psi$ are invariant. A particularly useful transformation is

$$y = \lambda^a y',\tag{3.36}$$

$$x = \lambda^b x',\tag{3.37}$$

$$\psi = \lambda^c \psi'.\tag{3.38}$$

Requiring the invariance of the equation (3.33) with respect to the transformation (3.36) to (3.38) gives

$$\frac{\partial \psi'}{\partial y'}\frac{\partial^2 \psi'}{\partial x'\partial y'} - \frac{\partial \psi'}{\partial x'}\frac{\partial^2 \psi'}{\partial (y')^2} = \lambda^{-c-a+b}\nu\frac{\partial^3 \psi'}{\partial (y')^3},\tag{3.39}$$

and performing the same for the boundary conditions gives

$$\lambda^{c-a}\frac{\partial \psi'}{\partial y'} = 0 \text{ for } y' = 0 \text{ and } x' > 0,\tag{3.40}$$

$$\lambda^{c-b}\frac{\partial \psi'}{\partial x'} = 0 \text{ for } y' = 0 \text{ and } x' > 0.\tag{3.41}$$

The requirement of invariance in the equations (3.39) to (3.41) leads to the algebraic relations

$$\left.\begin{array}{r}-c-a+b=0 \\ c-a=0 \\ c=b\end{array}\right\}\ \ b = a/2\ \left.\vphantom{\begin{array}{r}-c-a+b=0 \\ c-a=0 \\ c=b\end{array}}\right\}\ c = a/2,\tag{3.42}$$

which suggest that we take the change of variables

$$\psi = UBx^{1/2}f(\eta) \text{ with } \eta = A\frac{y}{x^{1/2}}.\tag{3.43}$$

If we substitute the change of variables above into equation (3.33), we obtain an ordinary differential equation for $f(\eta)$, given by

$$-\frac{1}{2}UBf(\eta)f''(\eta) = A\nu f'''(\eta),\tag{3.44}$$

and we chose

$$B = U^{-1/2}\sqrt{2\nu}\tag{3.45}$$

so that

$$\eta = \left(\frac{U}{2\nu x}\right)^{1/2}y,\tag{3.46}$$

such that

$$\psi = (2\nu U x)^{1/2}f(\eta).\tag{3.47}$$

The change of variables given by equations (3.46) and (3.47) is analog to the change of variables used in the Rayleigh problem discussed in Capter 1, where instead of $x/U$ we have time $t$. The analogy is that the disturbance due to the plate spreads out into the stream at the rate given by the unsteady problem (Rayleigh problem), but at the same time it is swept downstream with the fluid. The Rayleigh problem in Chapter 1 can be used to give an approximate solution to the problem here. The expression for the flow velocity $u$ for the Rayleigh problem can be used to estimate the downstream velocity relative to the plate by identifying $t$ with $x/U$. This is equivalent to the following approximations in the momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Firstly, the convection terms on the left are replaced by the approximation $U\partial u/\partial x$, and, secondly, the term $\nu\partial^2 u/\partial x^2$ is neglected in the viscous terms on the right. In

this way, we obtain the diffusion equation for $u$ with $t$ replaced by $x/U$. The boundary layer approximation retains the convection terms in full and makes only the second simplification. The Rayleigh approximation obviously overestimates the convection effects; hence its prediction of the boundary layer thickness will be too small and the value of the shear stress to great.

The equation that $f(\eta)$ has to satisfy is

$$ff'' + f''' = 0 \tag{3.48}$$

with boundary conditions:

$$f'(\eta) = 0 \text{ at } \eta = 0 \tag{3.49}$$

$$f(\eta) = 0 \text{ at } \eta = 0 \tag{3.50}$$

$$f'(\eta) = 1 \text{ at } \eta \to \infty \tag{3.51}$$

This is a boundary value problem for the function $f(\eta)$ which has no closed form solution, so we need to solve it numerically. Solving boundary value problems numerically is not an easy task. We would like to reduce this boundary value problem to an initial value problem. For the equation (3.48) this is possible. If $F(\eta)$ is any solution of equation (3.48), also

$$f = CF(C\eta) \tag{3.52}$$

is a solution, with $C$ an arbitrary constant of homology. Then,

$$\lim_{\eta \to \infty} f'(\eta) = C^2 \lim_{\eta \to \infty} F'(C\eta) = C^2 \lim_{\eta \to \infty} F'(\eta). \tag{3.53}$$

Since $f'(\eta) = 1$ for $\eta \to \infty$, we have that

$$C = \left\{ \lim_{\eta \to \infty} F'(\eta) \right\}^{-1/2} \tag{3.54}$$

We know from equation (3.52) that $f''(0) = C^3 F''(0)$, and if we specify $F''(0) = 1$, we have from equation (3.54) that

$$f''(0) = \left\{ \lim_{\eta \to \infty} F'(\eta) \right\}^{-3/2}. \tag{3.55}$$

To obtain the value $f''(0)$, we need to evaluate numerically the initial value problem for $F(\eta)$, given by equation (3.48) with initial conditions $F(0) = F'(0) = 0$ and $F''(0) = 1$, for a large value of $\eta$ to obtain $F'(\eta)$ as $\eta \to \infty$ and $f''(0)$ (see equation (3.55). Therefore, the initial value problem for $f(\eta)$ equivalent to the boundary value problem given by equations (3.48) to (3.51) is given by the ordinary differential equation (3.48) plus the initial conditions

$$f(0) = 0, \tag{3.56}$$

$$f'(0) = 0, \tag{3.57}$$

$$f''(0) = \left\{ \lim_{\eta \to \infty} F'(\eta) \right\}^{-3/2}. \tag{3.58}$$

Below we illustrate the result of the numerical integration of the initial value problem given by equation (3.48) and initial conditions (3.56) to (3.58) in the figures 1. We obtained numerically that $f''(0) \approx 0.46960$.
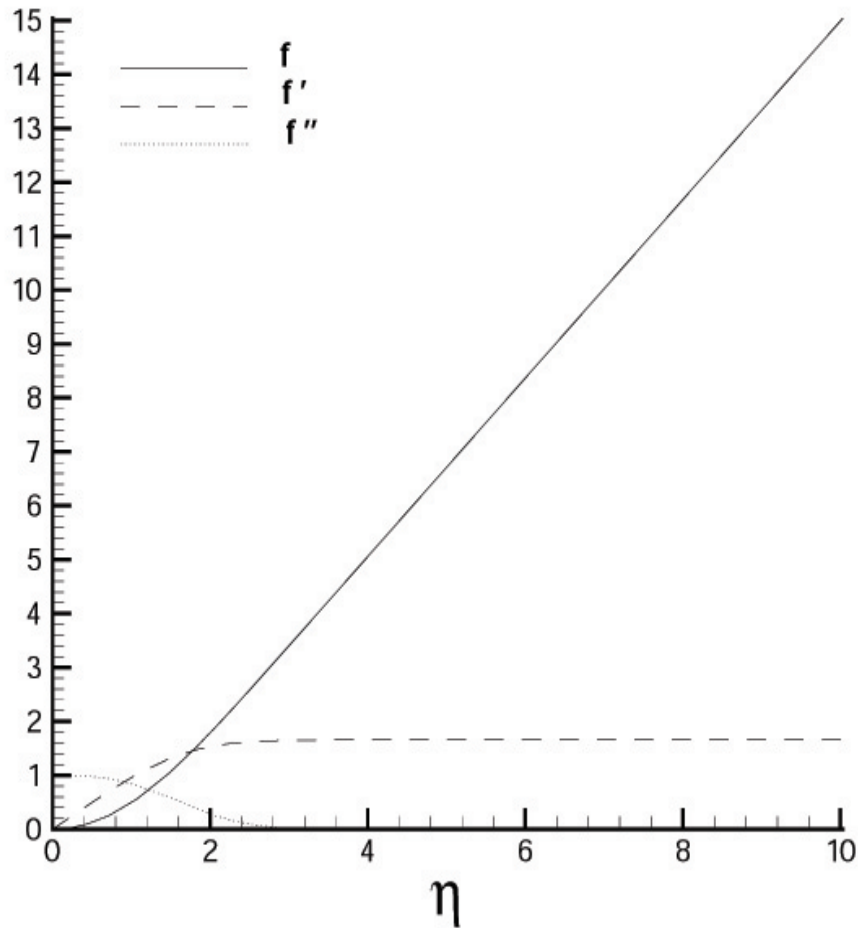
Figure 1: Functions $f(\eta), f'(\eta)$ and $f''(\eta)$. The horizontal axis represents the range of values of $\eta$ considered, and in the vertical axis we have the values of the functions $f(\eta), f'(\eta)$ and $f''(\eta)$.

Once we have computed numerically $f(\eta)$ and its derivatives up to second order, we can obtain the velocity components $(u, v)$ at any point $(x, y)$ of the flow domain, according to the equations

$$u = \frac{\partial \psi}{\partial y} = U f'(\eta), \tag{3.59}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left(\frac{U\nu}{2x}\right)^{1/2} f(\eta) - \frac{1}{2} U y f'(\eta). \tag{3.60}$$

### 3.1.1 Vorticity.

The vorticity of the flow in Cartesian coordinates in term of dimensional variables is given by

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{3.61}$$

If we non-dimensionalize the vorticity, according to

$$\omega = \frac{U}{L} \omega', \tag{3.62}$$

the non-dimensional form of the equation (3.61) is

$$\frac{U}{L}\omega' = \frac{U}{\delta}\frac{\partial u'}{\partial y'} - \frac{U\delta}{L^2}\frac{\partial v'}{\partial x'}, \tag{3.63}$$

and since $\delta \sim O(R^{-1/2}L)$, we have that

$$\omega' = R^{1/2}\frac{\partial u'}{\partial y'} - R^{-1/2}\frac{\partial v'}{\partial x'}, \tag{3.64}$$

and in the limit $R \to \infty$ we have

$$\omega' \sim R^{1/2}\frac{\partial u'}{\partial y'}. \tag{3.65}$$

In the context of the boundary-layer approximation, the vorticity in terms of dimensional variables is given by

$$\omega = \mu\frac{\partial u}{\partial y}. \tag{3.66}$$

We can write the vorticity for the Blasius boundary layer similarity solution by placing equation (3.47) for $\psi$ into the equation (3.66), which gives

$$\omega = \mu \frac{\partial^2 \psi}{\partial y^2} = \mu \left( \frac{U^3}{2\nu x} \right)^{1/2} f''(\eta) \tag{3.67}$$

### 3.1.2   Stress.

The normal components of the stress perpendicular and parallel to the flat plate expressed non-dimensionally are

$$\frac{\tau_{xx}}{\rho U^2} = - \frac{p}{\rho U^2} + 2 \underbrace{\frac{\nu}{U^2} \frac{\partial u}{\partial x}}_{O(1/R)}, \tag{3.68}$$

$$\frac{\tau_{yy}}{\rho U^2} = - \frac{p}{\rho U^2} + 2 \underbrace{\frac{\nu}{U^2} \frac{\partial v}{\partial y}}_{O(1/R)}. \tag{3.69}$$

Therefore, in the limit $R \to \infty$, we have that

$$\tau_{xx} = \tau_{yy} = -p \tag{3.70}$$

The shearing stress over surfaces parallel to the wall is

$$\tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \tag{3.71}$$

which is approximated in the same way as the vorticity, as follows.

$$\rho U^2 \tau'_{xy} = \rho \nu \left( \underbrace{\frac{U}{\delta}}_{\frac{U}{L} R^{1/2}} \frac{\partial u'}{\partial y'} + \underbrace{\frac{U\delta}{L^2}}_{\frac{U}{L} R^{-1/2}} \frac{\partial v'}{\partial y'} \right)$$

$$= \rho U^2 R^{-1/2} \frac{\partial u'}{\partial y'} + \rho U^2 R^{-3/2} \frac{\partial v'}{\partial x'} \tag{3.72}$$

In the limit $R \to \infty$, the shear stress is given by

$$\tau'_{xy} \sim R^{-1/2} \frac{\partial u'}{\partial y'} \tag{3.73}$$

or in terms of dimensional variables

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}. \tag{3.74}$$

For the Blasius laminar boundary layer similarity solution given by equation (3.47), the shear stress $\tau_{xy}$ is given by

$$\tau_{xy} \sim \mu \frac{\partial^2 \psi}{\partial y^2} = \mu \left( \frac{U^3}{2\nu x} \right)^{1/2} f''(\eta), \tag{3.75}$$

# 4 Boundary-layer Thickness, Skin friction, and Energy dissipation.

According to equation (2.22), the pressure across the boundary layer is constant in the boundary-layer approximation, and its value at any point is therefore determined by the corresponding main-stream conditions. If $U(x,t)$ now denotes the main stream velocity, so that

$$-\frac{1}{\rho} \frac{dp}{dx} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x}. \tag{4.76}$$

Elimination of the pressure from equation (2.21) gives in terms of dimensional variables the boundary layer momentum equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \tag{4.77}$$

and from equation (2.23), we have the mass conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.78}$$

In most physical problems the solutions of the boundary layer equations (4.77) and (4.78) are such that the velocity component $u$ attains its main-stream value $U$ only asymptotically as $R^{1/2}y/L \to \infty$. The thickness of the layer is therefore indefinite, as there is always some departure from the asymptotic value at any finite distance $y$ from the surface. In practice the approach to the limit is rapid and a point is soon reached beyond which the influence of viscosity is imperceptible. It would therefore be possible to regard the boundary layer thickness as a distance $\delta$ from the surface beyond which $u/Y > 0.99$, for example, but this is not sufficiently precise (since $\partial u/\partial y$ is small there) for experimental work, and is not of theoretical significance.

The scale of the boundary layer thickness can, however, be specified adequately by certain lengths capable of precise definition, both for experimental measurement and for theoretical study. These measures of boundary layer thickness are defined as follows:

- Displacement thickness $\delta_1$:

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \qquad (4.79)$$

- Momentum thickness $\delta_2$:

$$\delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad (4.80)$$

- Energy thickness $\delta_3$:

$$\delta_3 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy \qquad (4.81)$$

The upper limit of integration is taken as infinity owing to the asymptotic approach of $u/U$ to 1, but in practice the upper limit is the point beyond which the integrand is negligible.

$U\delta_1$ is the diminution, due to the boundary layer, of the volume flux across a normal to the surface; the streamlines of the outer flow are thus displaced away from the surface

through a distance $\delta_1$. Similarly, $\rho U^2 \delta_2$ is the flux of the defect of momentum, and $\frac{1}{2}\rho U^3 \delta_3$ is the flux of defect of kinetic energy.

Two other quantities related to these boundary layer thickness are the skin friction $\tau_\omega$ and the dissipation integral $D$. The skin friction is defined as the shearing stress exerted by the fluid on the surface over which it flows, and is therefore the value of $\tau_{xy}$ at $y = 0$, which by (3.74) is

$$\tau_\omega = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} \tag{4.82}$$

in terms of dimensional variables. The rate at which energy is dissipated by the action of viscosity has been shown to be $\mu \left( \frac{\partial u}{\partial y} \right)^2$ per unit time per unit volume, and $D$ is the integral of this across the layer:

$$D = \int_0^\infty \mu \left( \frac{\partial u}{\partial y} \right)^2 dy. \tag{4.83}$$

Consequently, $D$ is the total dissipation in a cylinder of small cross-section with axis normal to the layer per unit time per unit area of cross-section.

## 4.1  Quantities for the Blasius Boundary Layer Solution.

For the Blasius similarity solution for a two-dimensional boundary layer given by equation (3.47), we can compute the the quantities defined above:

- Displacement thickness $\delta_1$:

$$\begin{aligned}
\delta_1 &= \left( \frac{2\nu x}{U} \right)^{1/2} \int_0^\infty (1 - f'(\eta))d\eta \\
&= \left( \frac{2\nu x}{U} \right)^{1/2} \lim_{\eta \to \infty} (\eta - f(\eta))
\end{aligned} \tag{4.84}$$

- Momentum thickness $\delta_2$:

$$\delta_2 = \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^\infty f'(\eta)(1 - f'(\eta))d\eta$$

$$= \left(\frac{2\nu x}{U}\right)^{1/2} \left\{f(\eta)|_0^\infty - ff'|_0^\infty + \int_0^\infty ff''d\eta\right\}$$

and taking into account the boundary conditions (3.49) to (3.51), we obtain that

$$\delta_2 = \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^\infty ff''d\eta \qquad (4.85)$$

- Energy thickness $\delta_3$:

$$\delta_3 = \left(\frac{2\nu x}{U}\right)^{1/2} \int_0^\infty f'(\eta)(1 - f'(\eta)^2)d\eta$$

$$= \left(\frac{2\nu x}{U}\right)^{1/2} \left\{f(\eta)|_0^\infty - f(f')^2|_0^\infty + 2\int_0^\infty ff'f''d\eta\right\}$$

and taking into account the boundary conditions (3.49) to (3.51), we obtain that

$$\delta_3 = \left(\frac{2\nu x}{U}\right)^{1/2} 2\int_0^\infty ff'f''d\eta \qquad (4.86)$$

- Skin friction $\tau_\omega$:

$$\tau_{omega} = \mu \left(\frac{U^3}{2\nu x}\right)^{1/2} f''(\eta)|_{\eta=0},$$

and according to the initial condition $f''(0) = 1$, we have that

$$\tau_{omega} = \mu \left(\frac{U^3}{2\nu x}\right)^{1/2} \qquad (4.87)$$

- Dissipation integral $D$:

$$D = \mu \left(\frac{U^3}{2\nu x}\right) \int_0^\infty f''(eta)^2 d\eta \qquad (4.88)$$

# 5    Momentum and Energy Equations.

The skin friction and dissipation are connected with the boundary-layer thickness by two equations which represent the balance of momentum and energy within a small section of the boundary layer.

## 5.1    Momentum Integral

We can write equation (4.77) in the form

$$-\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial t}(u - U) + U\frac{\partial U}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}, \tag{5.89}$$

and if we multiply equation (4.78) by $(u - U)$ we obtain

$$(U - u)\frac{\partial u}{\partial x} + (U - u)\frac{\partial v}{\partial y} = 0. \tag{5.90}$$

If we add these two equations, we obtain

$$-\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial t}(u - U) + \frac{\partial}{\partial x}(Uu - u^2) + (U - u)\frac{\partial U}{\partial x} + \frac{\partial}{\partial x}(vU - vu). \tag{5.91}$$

Next, we integrate with respect to $y$ from 0 to $\infty$. This yields

$$-\nu\frac{\partial u}{\partial y}\big|_{y=0} = \frac{\partial}{\partial t}\int_0^\infty (u - U)dy + \frac{\partial}{\partial x}\int_0^\infty (Uu - u^2)dy + \frac{\partial U}{\partial x}\int_0^\infty (U - u)dy + (vU)|_{y=0}, \tag{5.92}$$

since $\partial u/\partial y$ and $v(U - u)$ tend to zero as $y \to \infty$. If there is no suction at the body surface, $v_{y=0} = 0$. We assume that there is no suction at the body surface, and by taking into account equations (4.79) to (4.82), we can write equation (5.92) in the form

$$\frac{\tau_\omega}{\rho} = \nu\frac{\partial u}{\partial y}\big|_{y=0} = \frac{\partial}{\partial t}(U\delta_1) + \frac{\partial}{\partial x}(U^2\delta_2) + U\frac{\partial U}{\partial x}\delta_1 \tag{5.93}$$

## 5.2 Energy Integral.

We multiply equation (4.77) and (4.78) respectively by $2u$ and $(U^2 - u^2)$, so we have

$$(4.77) \rightarrow -2u\nu\frac{\partial^2 u}{\partial y^2} = 2u\frac{\partial}{\partial t}(U - u) + 2uU\frac{\partial U}{\partial x} - 2u^2\frac{\partial u}{\partial x} - 2vu\frac{\partial u}{\partial y}, \qquad (5.94)$$

$$(4.77) \rightarrow 0 = (U^2 - u^2)\frac{\partial u}{\partial x} + (U^2 - u^2)\frac{\partial v}{\partial y}, \qquad (5.95)$$

and by adding them we obtain

$$2\nu\left(\frac{\partial u}{\partial y}\right)^2 - 2\nu\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial t}(Uu - u^2) + U^2\frac{\partial}{\partial t}\left(1 - \frac{u}{U}\right) + \frac{\partial}{\partial x}(U^2 u - u^3) + \frac{\partial}{\partial y}(vU^2 - vu^2)$$

$$(5.96)$$

If we integrate the equation above with respect to $y$ from $0$ to $\infty$, and if we take into account equations (4.79) to (4.83), we obtain

$$\frac{2D}{\rho} = \frac{\partial}{\partial t}(U^2\delta_2) + U^2\frac{\partial}{\partial t}\delta_1 + \frac{\partial}{\partial x}(U^3\delta_3), \qquad (5.97)$$

since $v(U^2 - u^2)$ and $\partial u/\partial y$ both tend to zero as $y$ tends to infinite and the term $(vU^2)_{y=0}$ is zero because it is assumed that there is no suction on the plate. The energy integral may also be regarded as an equation for the "kinetic energy defect" $\frac{1}{2}\rho(U^2 - u^2)$ per unit volume, namely

$$\frac{\partial}{\partial t}\int_0^\infty \frac{1}{2}\rho(U^2 - u^2)dy + \frac{\partial}{\partial x}\int_0^\infty \frac{1}{2}\rho(U^2 - u^2)udy = D + \rho\frac{\partial U}{\partial t}\int_0^\infty (U - u)dy \quad (5.98)$$

# 6 Approximate Method Based on the Momentum Equation: Pohlhausen's Method.

One of the earliest and, until recently, most widely used approximate methods for the solution of the boundary layer equation is that developed by Pohlhausen. This method is based on the momentum equation of Kármán, which is obtained by integrating the

boundary layer equation (4.77) across the layer, as shown in section 5.1. In the case of steady flow over an impermeable surface, the momentum equation (5.93) reduces to

$$\frac{\tau_\omega}{\rho U^2} = \frac{d}{dx}\delta_2 + \frac{2\delta_2 + \delta_1}{U}\frac{dU}{dx} \tag{6.99}$$

where $\tau_\omega = \mu(\partial u/\partial y)|_{y=0}$ is the skin friction, $\delta_1 = \int_0^\infty (1 - u/U)dy$ is the displacement thickness, and $\delta_2 = \int_0^\infty (u/U)(1 - u/U)dy$ is the momentum thickness of the boundary layer. The boundary conditions for the boundary layer equations (4.77) and (4.78) are

$$u = v = 0 \text{ at } y = 0, \tag{6.100}$$

$$u \to U(x) \text{ as } y \to \infty. \tag{6.101}$$

From the boundary layer equations (4.77) and (4.78) and their derivatives with respect to $y$, a set of conditions on $u$ can be derived with the aid of the boundary conditions (6.100) and (6.101). These conditions are

$$\text{From (6.100)} \to u = 0 \text{ at } y = 0, \tag{6.102}$$

$$\text{From (4.77) and (6.100)} \to \frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx} \text{ at } y = 0, \tag{6.103}$$

$$\text{From } \frac{\partial}{\partial y}(4.77), (4.78) \text{ and } (6.100) \to \frac{\partial^3 u}{\partial y^3} = 0 \text{ at } y = 0, \tag{6.104}$$

$$\text{From } \frac{\partial^2}{\partial y^2}(4.77), \frac{\partial}{\partial y}(4.78) \text{ and } (6.100) \to \frac{\partial^4 u}{\partial y^4} = \frac{1}{\nu}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x} \text{ at } y = 0, \tag{6.105}$$

$$\cdots\cdots,$$

and at $y \to \infty$ we have

$$u \to U \Rightarrow \frac{\partial u}{\partial y} \to 0 \Rightarrow \ldots \Rightarrow \frac{\partial^{(n)} u}{\partial y^{(n)}} \to 0 \Rightarrow \ldots \tag{6.106}$$

In the Pohlhausen's method, and similar approximate methods, a form for the velocity profile $u(x, y)$ is sought which satisfies the momentum equation (6.99) and some of the boundary condition (6.102) to (6.106). It is hoped that this form will approximate to

the exact profile, which satisfies all the conditions (6.102) to (6.106) as well as (6.99). The form assumed is

$$\frac{u}{U} = f(\eta) \text{ with } \eta = \frac{y}{\delta(x)}, \tag{6.107}$$

where $\delta(x)$ is the effective total thickness of the boundary layer. The function $f(\eta)$ may also depend on $x$ through certain coefficients which are chosen so as to satisfy some of the conditions (6.102) to (6.106). Altough strictly the conditions at infinity are only approached asymptotically, it is assumed that these conditions can be transfered from infinity to $y = \delta$ without appreciable error. Thus equations (6.102) to (6.106) become

$$f(0) = 0 \tag{6.108}$$

$$f''(0) = -\Lambda = \frac{\delta^2}{\nu} \frac{dU}{dx} \tag{6.109}$$

$$f'''(0) = 0 \tag{6.110}$$

$$f^{(iv)}(0) = \left\{ \frac{\delta^3}{\nu} f' \frac{d}{dx} \left( U \frac{f'}{\delta} \right) \right\} |_{\eta=0} \tag{6.111}$$

$$\dots\dots$$

$$f(1) = 1 \text{ and } f'(1) = f''(1) = \dots = f^{(n)}(1) = \dots = 0 \tag{6.112}$$

For the assumed velocity profile (6.107), we have that

$$\frac{\tau_\omega}{\rho U^2} = \frac{\nu}{\delta U} f'(0), \tag{6.113}$$

$$\delta_1 = \delta \int_0^1 (1 - f) d\eta, \tag{6.114}$$

$$\delta_2 = \delta \int_0^1 f(1 - f) d\eta, \tag{6.115}$$

If the form assumed for $f(\eta)$ involves $m$ unknown coefficients, these can be specified using $m$ of the boundary conditions (6.108) to (6.112) and the remaining unknown $\delta$ can be determined from the momentum equation (6.99). Also, if only the conditions (6.108), (6.109) and (6.110) are considered, $\tau_\omega \delta / \mu U$, $\delta_1/\delta$ and $\delta_2/delta$ are functions of

$\Lambda$ alone. In this case, substitution of relations (6.113) to (6.115) into (6.99) leads to an equation of the form

$$\frac{\nu}{U\delta}f'(0) = \frac{d}{dx}\left\{\delta\int_0^1 f(1-f)d\eta\right\} + \frac{2\delta}{U}\frac{dU}{dx}\int_0^1 f(1-f)d\eta + \frac{\delta}{U}\frac{dU}{dx}\int_0^1 (1-f)d\eta$$

$$\frac{\nu}{U\delta}f'(0) = \frac{d\delta}{dx}\int_0^1 f(1-f)d\eta + \frac{\delta}{U}\frac{dU}{dx}\left\{2\int_0^1 f(1-f)d\eta + \int_0^1 (1-f)d\eta\right\}$$

$$\frac{f'(0)}{U} = \frac{\delta}{\nu}\frac{d\delta}{dx}\frac{dU}{dx}\frac{\int_0^1 f(1-f)d\eta}{\frac{dU}{dx}} + \frac{1}{2}\frac{\delta^2}{\nu}\frac{d^2U}{dx^2}\frac{\int_0^1 f(1-f)d\eta}{\frac{dU}{dx}} - \frac{1}{2}\frac{\delta^2}{\nu}\frac{d^2U}{dx^2}\frac{\int_0^1 f(1-f)d\eta}{\frac{dU}{dx}}$$

$$+ \frac{1}{U}\frac{\delta^2}{\nu}\frac{dU}{dx}\left\{2\int_0^1 f(1-f)d\eta + \int_0^1 (1-f)d\eta\right\} \tag{6.116}$$

Since $\Lambda = (\delta^2/\nu)dU/dx$, we can write the equation above in the form

$$\frac{d\Lambda}{dx} = \frac{1}{U}\frac{dU}{dx}\left\{\frac{f'(0)}{\int_0^1 f(1-f)d\eta} - \Lambda\left[2 - \frac{\int_0^1 (1-f)d\eta}{\int_0^1 f(1-f)d\eta}\right]\right\} + \frac{\frac{d^2U}{dx^2}}{\frac{dU}{dx}}\frac{\Lambda}{2} \tag{6.117}$$

or simply

$$\frac{d\Lambda}{dx} = \frac{1}{U}\frac{dU}{dx}g(\Lambda) + \frac{\frac{d^2U}{dx^2}}{\frac{dU}{dx}}h(\Lambda) \tag{6.118}$$

where

$$g(\Lambda) = \left\{\frac{f'(0)}{\int_0^1 f(1-f)d\eta} - \Lambda\left[2 - \frac{\int_0^1 (1-f)d\eta}{\int_0^1 f(1-f)d\eta}\right]\right\} \tag{6.119}$$

$$h(\Lambda) = \Lambda/2 \tag{6.120}$$

Pohlhausen used a family of velocity profiles given by the quartic polynomial

$$u/U = f(\eta) = 2\eta - 2\eta^3 + \eta^4 + \frac{1}{6}\Lambda\eta(1-\eta)^3 \tag{6.121}$$

chosen to satisfy the boundary conditions

$$f(0) = 0, f''(0) = -\Lambda, f(1) = 1, f'(1) = f''(1) = 0$$

With the chosen velocity profile (6.121), we can evaluate the functions $g(\Lambda)$ and $h(\Lambda)$, and then integrate numerically equation (6.118) to obtain $\Lambda$ as a function of $x$ and through equation $\delta^2 = \nu\Lambda/(dU/dx)$. By the substitution of $\Lambda = \Lambda(x)$ and $\delta = \delta(x)$ in the assumed velocity profile $f(\eta)$, we obtain the velocity profile for any $x$ and $y$.

# 7 Stagnation Point flow.

For an ideal fluid the flow against an infinite flat plate in the plane $y = 0$ is given by

$$u = Ux, \tag{7.122}$$

$$v = -Uy, \tag{7.123}$$

where $U$ is a constant. When viscosity is included, it still must be true that $u$ is proportional to $x$, for small $x$ and for all $y$. Thus, for small $x$, at least, we may take

$$u = kxF(y) \tag{7.124}$$

The governing equations for steady flow in terms of dimensional variables are given by equations (2.1) to (2.3). If we substitute equation (7.124) into the continuity equation (2.3), we obtain

$$\frac{\partial v}{\partial y} = -kF(y) \tag{7.125}$$

This suggest that we take

$$u = kxf'(Ay), \tag{7.126}$$

$$v = -Bf(Ay). \tag{7.127}$$

Now, we substitute equations (7.126) and (7.127) into the governing equations (2.1) to (2.3). We obtain from

$$(2.1) \rightarrow k^2 x f'(Ay)^2 - Bf(Ay)kx f''(Ay)A = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu kx A^2 f'''(Ay), \qquad (7.128)$$

$$(2.2) \rightarrow B^2 f(Ay)Af'(Ay) = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu Bf''(Ay)A^2, \qquad (7.129)$$

$$(2.3) \rightarrow kf'(Ay) - BAf'(Ay) = 0. \qquad (7.130)$$

This last equation implies that $BA = k$, so we can write equations (7.128) and (7.130) in the form

$$(7.128) \rightarrow k^2 x (f')^2 - k^2 x f'' f = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu kx A^2 f''' \qquad (7.131)$$

$$(7.129) \rightarrow Bkff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu kA f'' \qquad (7.132)$$

We can solve equation (7.131) for the pressure derivative with respect to $x$ to obtain

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = k^2 x[(f')^2 - ff'' - \frac{\nu}{k}A^2 f'''] \qquad (7.133)$$

Since the the term between brackets in the right side of the equation above is not a function of $x$, we set

$$(f')^2 - ff'' - \frac{\nu}{k}A^2 f''' = 1, \qquad (7.134)$$

which implies, according to equation (7.133), that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = k^2 x \rightarrow -\frac{1}{\rho}p = \frac{1}{2}k^2 x^2 + C(y) \qquad (7.135)$$

If we substitute equation (7.135) into equation (7.132), we have the relation

$$Bkff' = \frac{dC}{dy} - \nu kA f'', \qquad (7.136)$$

and if we integrate this equation, we obtain that

$$C(y) = \frac{1}{2}B\frac{k}{A}f^2 + \nu k f' - C_0. \tag{7.137}$$

Next, we substitute the expression for $C(y)$ above into the equation (7.135) for the pressure, which gives

$$-\frac{1}{\rho}p = \frac{1}{2}k^2x^2 + \frac{1}{2}B\frac{k}{A}f^2 + \nu k f' - C_0. \tag{7.138}$$

To simplify equation (7.134), we chose the coefficient of $f'''$ equal to one, which implies that

$$\nu\frac{A^2}{k} = 1 \rightarrow A = \sqrt{\frac{k}{\nu}}, \tag{7.139}$$

and since $BA = k$, we have that

$$B = \sqrt{\nu k}. \tag{7.140}$$

With equations (7.138) and (7.139), we can write the equation (7.138) as follows

$$\frac{p_0 - p}{\rho} = \frac{1}{2}k^2x^2 + \frac{1}{2}\nu k f^2 + \nu k f' \tag{7.141}$$

with $C_0 = p_0/\rho$ and $p_0$ is the pressure as $y \rightarrow \infty$. The function $f(\eta)$, where $\eta = \sqrt{k/\nu}y$ satisfies the ordinary differential equation

$$(f')^2 - ff'' - f''' - 1 = 0 \tag{7.142}$$

with boundary conditions:

- No slip condition at the plate surface.

$$u = 0 \text{ at } y = 0 \rightarrow f'(\eta) = 0 \text{ at } \eta = 0, \tag{7.143}$$

- No flux across the plate:

$$v = 0 \text{ at } y = 0 \rightarrow f(\eta) = 0 \text{ at } \eta = 0, \qquad (7.144)$$

- At $y \rightarrow \infty$, we have $u = kx$, which implies that

$$f'(\eta) = 1 \text{ as } \eta \rightarrow \infty \qquad (7.145)$$

In summary, the function $f(\eta)$ is the solution of the boundary value problem given by the equations (7.142) to (7.145), which has no closed form solution. The equation ordinary differential equation (7.142) is non-linear and has to be solved numerically together with the boundary conditions (7.143) to (7.145). Figure 2 below illustarte the result of the numerical evaluation of the boundary value problem given by equations (7.142) to (7.145) for $\eta$ in the range $0 < \eta < 10$.
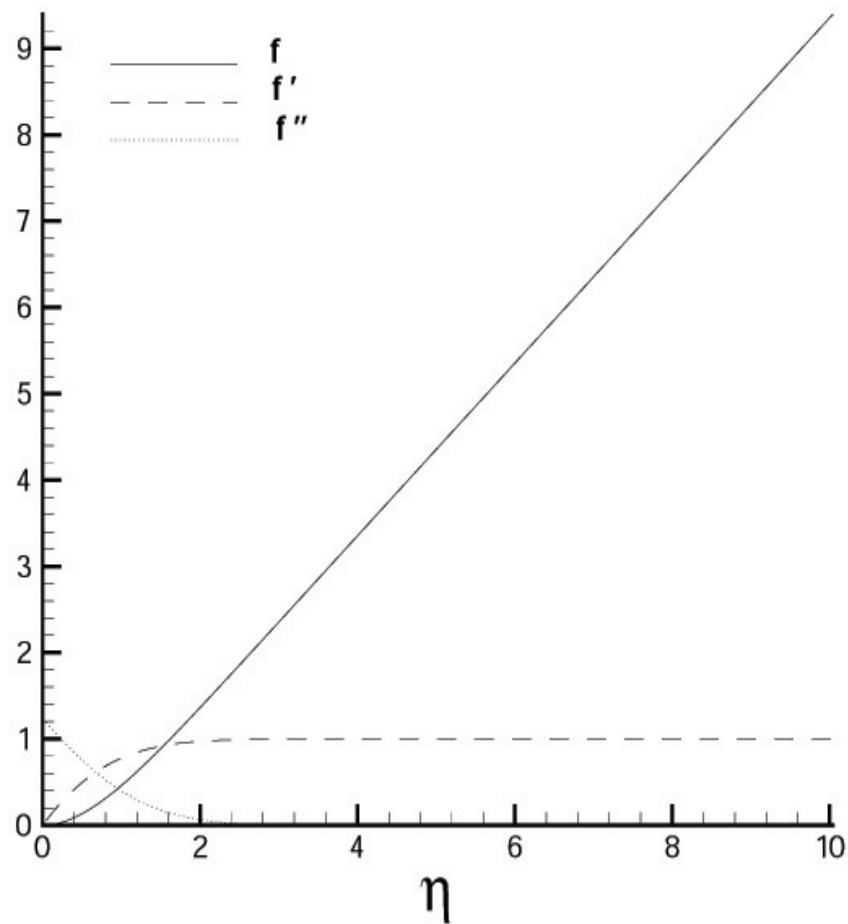
Figure 2: Functions $f(\eta), f'(\eta)$ and $f''(\eta)$. The horizontal axis represents the range of values of $\eta$ considered, and in the vertical axis we have the values of the functions $f(\eta), f'(\eta)$ and $f''(\eta)$.

Once we know the values of $f(\eta)$, $f'(\eta)$ and $f''(\eta)$, we can obtain the velocities $u$ and $v$ given, respectivelly, by equations (7.126) and (7.127) with $B$ and $A$ given, respectively, by equations (7.139) and (7.140), the pressure field is given by equation (7.141).

# 8 Two-dimensional Laminar Jet.

We consider a two-dimensional jet as illustrated in the figure below. $x$ is the horizontal coordinate and $y$ is the vertical coordinate. $u$ and $v$ are, respectively, the horizontal and vertical fluid velocities. The jet in the direction of the $x$ axis generates a flow where the fluid velocity along the $y$ axis tends to zero. We assume that the boundary layer approximation is valid and the governing equation for the fluid motion are equations (2.21) to (2.23), but with $\partial u/\partial t$. The pressure does not vary in the $y$ direction according to equation (2.22), so the pressure is constant across the boundary layer and its gradient is given by the pressure gradient outside the boundary layer. For this problem there is no pressure gradient, so the governing equations for the fluid motion are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{8.146}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8.147}$$

Next, we integrate equation (8.146) with respect to the $y$ variable from $-\infty$ to $+\infty$, which gives

$$\int_{-\infty}^{\infty} (u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y})dy = \nu\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2}dy \tag{8.148}$$

We can write

$$v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(vu) - u\frac{\partial v}{\partial y},$$

and if we multiply the continuity equation (8.147) by $u$, we can write the equation above as

$$v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(vu) - u\frac{\partial u}{\partial x}, \tag{8.149}$$

Next, we substitute equation (8.149) into equation (8.148), we perform the integration and we use the boundary condition

$$u \to 0 \text{ as } y \to \pm\infty. \tag{8.150}$$

Equation (8.148) simplifies to

$$\frac{\partial}{\partial x} \int_{-\infty}^{+\infty} u^2 dy = 0 \tag{8.151}$$

Thus, the momentum flux is constant in $x$. In other words,

$$\rho \int_{-\infty}^{\infty} u^2 dy = M \tag{8.152}$$

We call $\delta(x)$ the order of the magnitude of the boundary layer thickness at position $x$. Since the orifice is very small we have $\delta(0) \to 0$. At the orifice the equation (8.152) can be written as

$$\rho u(0)^2 \delta(0) = M, \tag{8.153}$$

which implies that

$$u(0) = \delta(0)^{-1/2} \tag{8.154}$$

The Mass flux at the orifice is

$$\rho u(0)\delta(0) \sim \delta(0)^{1/2} \to 0, \tag{8.155}$$

hence unimportant. A jet is the result of a momentum source, not a volume source. Next, we are going to solve the boundary layer equations for the jet. We introduce the stream function $\psi$ related to the velocities $u$ and $v$ according to the equations

$$u = \frac{\partial \psi}{\partial y} \tag{8.156}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{8.157}$$

In terms of the stream function, the $x$ momentum equation (8.146) assume the form

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} \tag{8.158}$$

with the boundary conditions

$$\psi \to 0 \text{ as } y \to \pm\infty \tag{8.159}$$

and

$$\rho \int_{-\infty}^{\infty} \left(\frac{\partial \psi}{\partial y}\right)^2 dy = M \tag{8.160}$$

We are going to solve the boundary value problem given by equations (8.158) to (8.160) by looking for a similarity solution. We look for a one-parameter transformation of variables $y, x$ and $\psi$ under which the equations of the boundary value problem mentioned above are invariant. A particular useful transformation is

$$x = \lambda^a x', y = \lambda^b y' \text{ and } \psi = \lambda^c \psi' \tag{8.161}$$

The requirement of invariance of the boundary value problem (8.158) to (8.160) under such transformation implies that

$$c + b - a = 0 \tag{8.162}$$
$$c - b = 0 \tag{8.163}$$

and no information is gained from equation (8.160). From the equations (8.162) and (8.163) we obtain

$$c = a/3 \text{ and } b = 2a/3 \tag{8.164}$$

This suggest that we take

$$\frac{\psi}{Bx^{1/3}} = f(\eta) \text{ with } \eta = \frac{Cy}{x^{2/3}}, \tag{8.165}$$

where the coefficients $B$ and $C$ are chosen to simplify the appearance of the final equation. We substitute (8.165) into equation (8.158), which gives the ordinary differential equation

$$3f''' + (f')^2 + ff'' = 0, \tag{8.166}$$

and for $\psi$ and $\eta$ we have

$$\psi = \left(\frac{M\nu x}{\rho}\right)^{1/3} f(\eta) \text{ with } \eta = \left(\frac{M}{\rho\nu^2 x^2}\right)^{1/3} \tag{8.167}$$

the boundary conditions become

$$f'(\pm\infty) \to 0, f(0) = f''(0) = 0(\text{symmetry}) \tag{8.168}$$

and

$$1 = \int_{-\infty}^{\infty} |f'(\eta)|^2 d\eta \tag{8.169}$$

Now we integrate equation (8.166) once with respect to $\eta$, which gives

$$3f'' + ff' = \text{ constant } = 0,$$

and integrating again

$$3f' + \frac{1}{2}f^2 = c^2 \tag{8.170}$$

We now write $f = \sqrt{2}F$ and $\eta = 3\sqrt{2}\zeta$. then equation (8.170) assumes the form

$$\frac{dF}{d\zeta} + F^2 = c^2 \rightarrow \frac{dF/c}{1 - (F/c)^2} = cd\zeta \tag{8.171}$$

which can be integrated in closed form, so we have

$$c\zeta = \tanh^{-1}(F/c) \tag{8.172}$$

since $F(0) = 0$. Thus

$$f = \sqrt{2}F = \sqrt{2}c\tanh\left(\frac{c\eta}{3\sqrt{2}}\right). \tag{8.173}$$

We substitute the equation above for $f$ in the boundary condition (8.169), which gives

$$1 = \frac{c^3\sqrt{2}}{3} \int_{-\infty}^{\infty} \text{sech}^4(c\zeta)d(c\zeta) = \frac{4\sqrt{2}c^3}{9} \rightarrow c^3 = \frac{9}{4\sqrt{2}} \tag{8.174}$$

Finally, we write

$$\chi = \left(\frac{M}{48\rho\nu^2}\right)^{1/3} \frac{y}{x^{2/3}} \tag{8.175}$$

The final solution has the form

$$\psi = \left(\frac{9M\nu x}{2\rho}\right)^{1/3} \tanh(\chi) \tag{8.176}$$

and for the velocities we have

$$u = \frac{\partial\psi}{\partial y} = \left(\frac{3M^2}{32\rho^2\nu x}\right)^{1/3} \text{sech}^2(\chi) \tag{8.177}$$

$$v = -\frac{\partial\psi}{\partial x} = \left(\frac{M\nu}{6\rho x^2}\right)^{1/3} (2\chi\text{sech}^2\chi - \tanh\chi) \tag{8.178}$$

Next, we discuss the implications of the results obtained above. The jet width can be defined by $\chi = \pm\chi_0$ such that $u \rightarrow 0$. Then from equation (8.177), we realize that

the jet width is proportional to $x^{2/3}$. The centerline velocity, let say $u_{max}$, according to equation (8.177) is proportional to $x^{-1/3}$. As $\chi \to \pm\infty$, according to equation (8.178), $v \to \mp(M\nu/6\rho x^2)^{1/3}$. This implies a that there is entrainment from the jet eddges as show in the figure 4. If we define the Reynolds number as $R = u_{max}\delta/nu$, equation (8.177) implies that $R \propto x^{1/3}$. To illustrate the streamlines for the flow generated by the jet, we present figure 3. The velocity field is illustrated in figure 3.
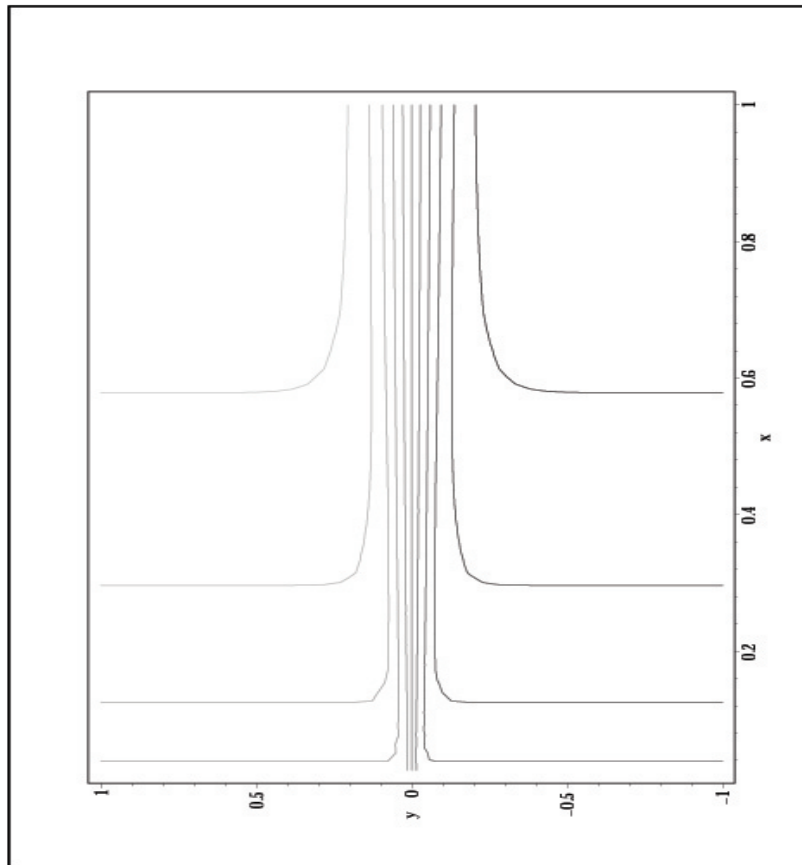


Figure 3: Streamlines obtained from equation (8.176) with $M = 1000kg/sec^2, \nu = 0.01m^2/sec$ and $\rho = 1000kg/m^3$.
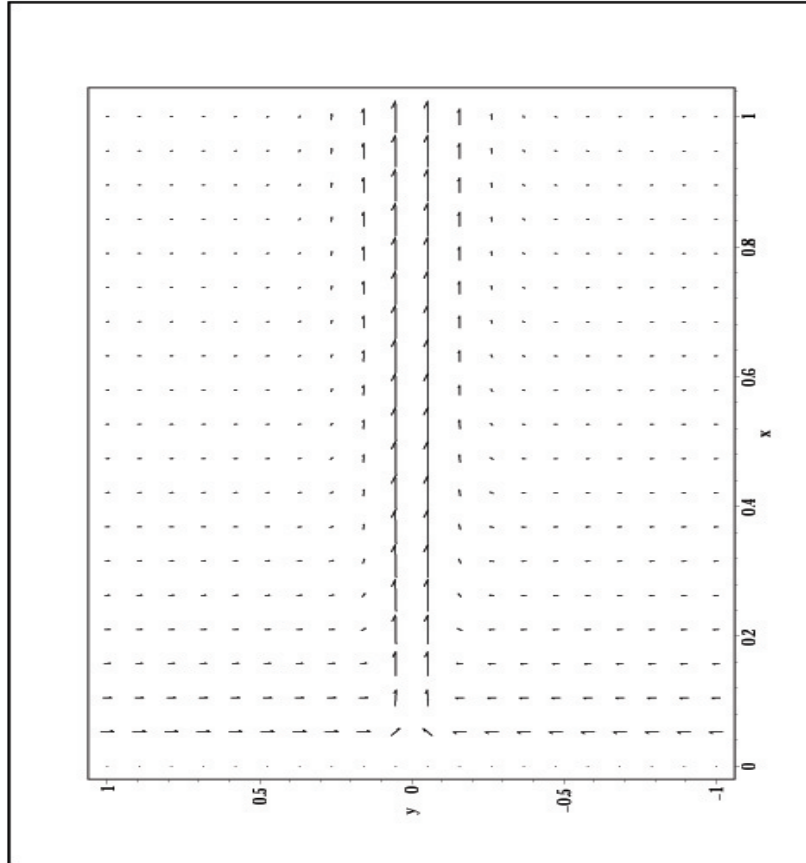
Figure 4: Velocity field obtained from equations (8.177) and (8.178) with $M = 1000kg/sec^2, \nu = 0.01m^2/sec$ and $\rho = 1000kg/m^3$.