

Notes on
1.63 Advanced Environmental Fluid Mechanics
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 3-impulsive.tex

3.8 Impulsive motion of a blunt body and tendency for separation

Ref: H. Schlichting, Boundary layer theory, p 400 ff.

As an example of unsteady boundary layer, let us consider the initial stage ($U_o T/L \ll 1$) of a boundary layer due to the impulsive start of motion near a blunt body, see the sketch in Figure 3.8.1.

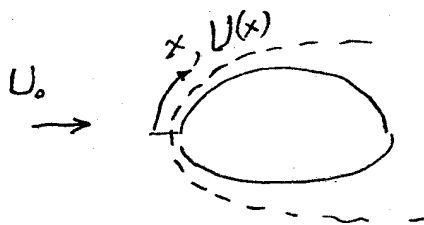


Figure 3.8.1: Boundary layer around a blunt body

Let us start with the boundary layer approximation and introduce a perturbation expansion in powers of the small ratio $U_o T/L$,

$$u = u^{(1)} + \left(\frac{U_o T}{L}\right) u^{(2)} + \left(\frac{U_o T}{L}\right)^2 u^{(3)} \dots, \quad (3.8.1)$$

$$p = p^{(1)} + \left(\frac{U_o T}{L}\right) p^{(2)} + \left(\frac{U_o T}{L}\right)^2 p^{(3)} + \dots \quad (3.8.2)$$

We then get

$$u_x^{(1)} + v_y^{(1)} + \left(\frac{U_o T}{L}\right) (u_x^{(2)} + v_y^{(2)}) + \dots = 0, \quad (3.8.3)$$

and

$$\begin{aligned} u_t^{(1)} + \frac{U_o T}{L} u_t^{(2)} + \frac{U_o T}{L} (u^{(1)} u_x^{(1)} + v^{(1)} u_y^{(1)}) + O\left(\frac{U_o T}{L}\right)^2 \\ = \frac{U_o T}{L} U U_x + u_{yy}^{(1)} + \frac{U_o T}{L} u_{yy}^{(2)} + O\left(\frac{U_o T}{L}\right)^2 \end{aligned} \quad (3.8.4)$$

$$(3.8.5)$$

Equating the coefficients of $\left(\frac{U_o T}{L}\right)^0$ we get the first (leading) order perturbation equations in normalized coordinates,

$$u_x^{(1)} + v_y^{(1)} = 0, \quad (3.8.6)$$

$$u_t^{(1)} = u_{yy}^{(1)} \quad (3.8.7)$$

subject to the initial conditions:

$$u^{(1)} = v^{(1)} = 0. \quad t = 0, \quad \forall y; \quad (3.8.8)$$

and the boundary condtions

$$u^{(1)} = v^{(1)} = 0. \quad y = 0, \quad \forall t; \quad (3.8.9)$$

$$u^{(1)} = U, \quad y \rightarrow \infty \quad (3.8.10)$$

Equating the coefficient of $\left(\frac{U_o T}{L}\right)$, we get the second order perturbation equations in normalized coordinates,

$$u_x^{(2)} + v_y^{(2)} = 0, \quad (3.8.11)$$

$$u_t^{(2)} + (u^{(1)} u_x^{(1)} + v^{(1)} u_y^{(1)}) = U U_x + u_{yy}^{(2)} + O\left(\frac{U_o T}{L}\right)^2 \quad (3.8.12)$$

subject to the same initial and boundary conditions on the wall as the first order problem, except that

$$u^{(2)} \rightarrow 0, \quad y \rightarrow \infty \quad (3.8.13)$$

To return to physical variables, we need only add the coefficient ν in front of the viscous stress term u_{yy} in (3.8.7), and (3.8.12). The first order problem for the tangential velocity is precisely the Rayleigh problem

$$u_t^{(1)} = u_{yy}^{(1)} \quad (3.8.14)$$

subject to the initial conditions:

$$u^{(1)} = 0. \quad t = 0, \quad \forall y; \quad (3.8.15)$$

and the boundary condtions

$$u^{(1)} = 0. \quad y = 0, \quad \forall t; \quad (3.8.16)$$

$$u^{(1)} = U, \quad y \rightarrow \infty \quad (3.8.17)$$

The solution is

$$u^{(1)}(x, y, t) = U(x)\text{erf}(\eta) = U(x)\frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta \quad (3.8.18)$$

where

$$\eta = \frac{y}{\sqrt{2\nu t}} \quad (3.8.19)$$

Integrating the continuity equation (3.8.6) we get

$$v^{(1)} = - \int_0^y \frac{\partial u_1}{\partial x} dy = - \frac{dU}{dx} 2\sqrt{\nu t} \int_0^\eta \text{erf}(\eta) d\eta \quad (3.8.20)$$

To simply the notation we introduce

$$\text{erf}(\eta) = \zeta'_0(\eta), \quad \int_0^\eta \text{erf}(\eta) d\eta = \zeta_0(\eta) \quad (3.8.21)$$

so that

$$u^{(1)} = U(x)\zeta'_0(\eta), \quad v^{(1)} = - \frac{dU}{dx} 2\sqrt{\nu t} \zeta_0(\eta) \quad (3.8.22)$$

The second-order approximation is

$$u_t^{(2)} - \nu u_{yy}^{(2)} = UU_x - u^{(1)}u_x^{(1)} - v^{(1)}u_y^{(1)} \quad (3.8.23)$$

subject to the initial and boundary conditions that

$$u^{(2)}(y, 0) = 0, \quad u^{(2)}(y, t) = 0 \quad \text{for } y = 0, \infty \quad (3.8.24)$$

The right hand side of (3.8.23) can be worked out so that

$$\begin{aligned} u_t^{(2)} - \nu u_{yy}^{(2)} &= UU_x \left[1 - (\text{erf}(\eta))^2 + e^{-\eta^2} \int_0^\eta \text{erf}(\eta) d\eta \right] \\ &= UU_x \left[1 - (h')^2 + hh'' \right] = UU_x F(\eta) \end{aligned} \quad (3.8.25)$$

A similarity solution is possible. Let us seek a one-parameter transformation,

$$u^{(2)} = \lambda^a u^{(2)'}, \quad t = \lambda^b t', \quad y = \lambda^c y'$$

From (3.8.23) we get

$$\lambda^{a-b} \frac{\partial u^{(2)'}}{\partial t'} - \nu \lambda^{a-2c} \frac{\partial^2 u^{(2)'}}{\partial y'^2} = UU_x F(\lambda^{c-b/2} \eta')$$

Note that x is just a parameter. Clearly $a = b = 2c$ so that we can take

$$\frac{u^{(2)}}{t} = f(\eta) UU_x \quad (3.8.26)$$

Substituting (3.8.26) into (3.8.25), we get a linear ordinary differential equation

$$f'' + 2\eta f' - 4f = 4 \left[(\zeta_0')^2 - \zeta_0 \zeta_0'' - 1 \right] \quad (3.8.27)$$

subject to the boundary conditions that

$$f = 0, \quad \eta = 0, \infty \quad (3.8.28)$$

The solution is not difficult (see Schlichting, eq. 15.43, p. 400).

$$\begin{aligned} f = & \operatorname{erfc}(\eta) \left[-\frac{3}{\sqrt{\pi}} e^{-\eta^2} + 2 - \left(\frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}} \right) + \frac{\sqrt{\pi}}{2} (2\eta^2 + 1) \right] \\ & + \frac{1}{2} (2\eta^2 - 1) \operatorname{erfc}^2(\eta) + \frac{2}{3} e^{-2\eta^2} \\ & + e^{-\eta^2} \left[\frac{\eta}{\sqrt{\pi}} - \frac{4}{3\pi} + \eta \left(\frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}} \right) \right] \end{aligned} \quad (3.8.29)$$

The solution is plotted in Figure 3.8.2.

The total solution is

$$u = U \operatorname{erf}(\eta) + t U U_x f(\eta) \quad (3.8.30)$$

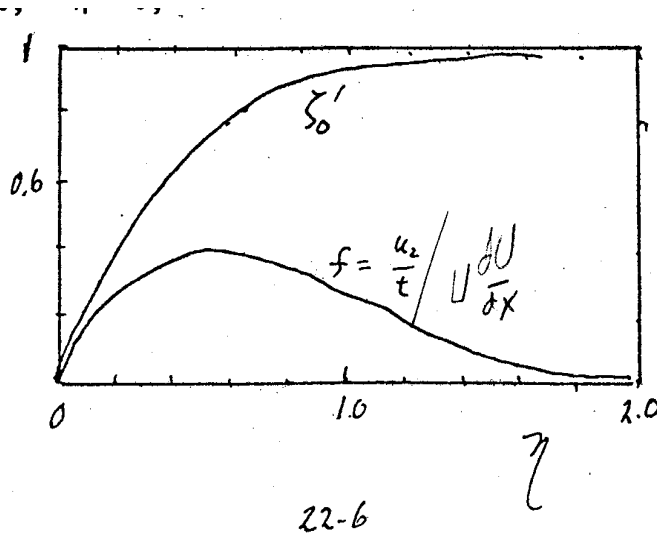


Figure 3.8.2: Solution to the problem of impulsive start.

Separation

For a given $U(x)$ when and where will separation first occur? Namely, when is

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = 0$$

Let us use (3.8.30) for a crude estimate. Since

$$\frac{\partial u}{\partial y} = [U(\operatorname{erf}\eta)' + UU_x t f'(\eta)] \frac{\partial \eta}{\partial y}$$

It can be shown that at $\eta = 0$,

$$(\operatorname{erf}\eta)' = \frac{2}{\sqrt{\pi}}, \quad f'(\eta) = \frac{2}{\sqrt{\pi}} \left(1 + \frac{4}{3\pi}\right)$$

It follows that

$$U + t_s \left(1 + \frac{4}{3\pi}\right) UU_x = 0$$

or

$$t_s = -\frac{0.7}{UU_x} \quad (3.8.31)$$

Note that $t_s > 0$ only for $U_x < 0$, i.e., a decelerated flow. This is a very crude and mathematically illegitimate estimate since we are equating two terms of different order.

Nevertheless let us apply this result to the impulsive flow passing a circular cylinder from the left. Let U_o be the constant velocity at infinity and the polar angle θ be measured from the upstream stagnation point, then $x = a\theta$ where a is the radius, see Figure 3.8.3. It is well known in the potential theory that the potential is

$$\phi = U_o \left(r + \frac{a^2}{r}\right) \cos(\pi - \theta)$$

The tangential velocity along the cylinder $r = a$ is

$$\frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{U_o}{r} \left(r + \frac{a^2}{r}\right) \sin(\pi - \theta), \quad r = a$$

or

$$U = 2U_o \sin(\pi - \theta) = 2U_o \sin(\theta) = 2U_o \sin x/a$$

The minimum t_s occurs at the rear stagnation point, $x = \pi a$ at which

$$t_s = \frac{0.35a}{U_o}, \quad \text{or} \quad \frac{U_o t_s}{a} = 0.35$$

Note that the last condition indicates the illegitimacy of this estimate. Nevertheless we use it here as an order-of-magnitude guide which may be improved by working out higher order terms.

In offshore structures, wave induced oscillatory flows around a pile can be separated and hence affect the drag force on the pile. As an order estimate we take $U_o = \omega A$ where ω = frequency and A = wave amplitude. Hence there is no separation if

$$\frac{\omega A t_s}{a} < 0.35, \quad \text{or} \quad \frac{A}{a} < \frac{0.35}{\omega t_s}$$

Since flow changes direction after every half period π/ω , there is no separation in every half period if

$$\frac{A}{a} < \frac{0.35}{\pi} = 0.1$$

This is of course very crude. Experimentally Keulegan and Carpenter have established that separation occurs in waves if A/a exceeds 1. The ratio A/a is now known as the Keulegan and Carpenter number.

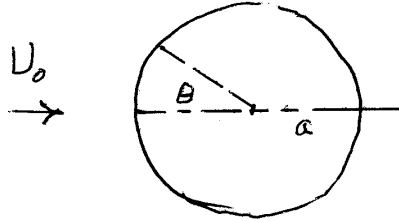


Figure 3.8.3: Definition of coordinates for a circular cylinder.