

Notes on  
**1.63 Advanced Environmental Fluid Mechanics**  
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October 11, 2002

3-8unsteadyBL.tex

### 3.8 Unsteady boundary layers

Let us begin from the full momentum equation

$$\vec{q}_t + \vec{q} \cdot \nabla \vec{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q} \quad (3.8.1)$$

Let the velocity and times scales be  $U_o$  and  $T$ , the tangential length scale be  $L$  and the transverse length scale be  $\delta \sim \sqrt{\nu T}$ . Hence the suitable normalization is

$$\begin{aligned} x' &= x/L, & y' &= y/\sqrt{\nu T}, & t' &= t/T, \\ u' &= u/U_o, & v' &= \frac{vL}{U_o\delta} = \frac{v}{U_o} \sqrt{\frac{L^2}{\nu T}}, \\ p &= \frac{pT}{\rho U_o L}, & U' &= U/U_o. \end{aligned} \quad (3.8.2)$$

If primes are omitted for brevity, the dimensionless equations are,

$$u_x + v_y = 0, \quad (3.8.3)$$

$$u_t + \frac{U_o T}{L} (uu_x + vv_y) = -p_x + \frac{\nu T}{L^2} u_{xx} + u_{yy} \quad (3.8.4)$$

$$\frac{\nu T}{L^2} \left[ v_t + \frac{U_o T}{L} (uv_x + vv_y) \right] = -p_y + \frac{\nu T}{L^2} \left[ \frac{\nu T}{L^2} v_{xx} + v_{yy} \right] \quad (3.8.5)$$

Outside the viscous boundary layer,

$$U_t + \left( \frac{U_o T}{L} \right) U U_x = -\frac{1}{\rho} p_x \quad (3.8.6)$$

Two parameters control the motion:  $U_o T/L$  (inertia) and  $\nu T/L^2$  (viscosity).

Several scenarios are possible:

1. Low amplitude and slow motion:  $U_o T/L \ll 1, \nu T/L^2 = O(1)$ . The tangential and transverse scales are comparable. To the leading order, the approximate equations in physical coordinates are

$$u_x + v_y = 0, \quad (3.8.7)$$

$$\vec{q}_t = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q} \quad (3.8.8)$$

This is just the Oseen's approximation.

2. Finite amplitude, fast motion,  $U_oT/L = O(1)$ ,  $\nu T/L^2 \ll O(1)$ . The boundary layer is thin. To the leading order, nonlinearity is important in the boundary layer.

$$u_x + v_y = 0, \quad (3.8.9)$$

$$u_t + \frac{U_oT}{L}(uu_x + vv_y) = -p_x + u_{yy} = U_t + \frac{U_oT}{L}UU_x + u_{yy} \quad (3.8.10)$$

or, in physical coordinates,

$$u_t + (uu_x + vv_y) = U_t + UU_x + \nu u_{yy} \quad (3.8.11)$$

3. Small-amplitude and fast motion.  $\nu T/L^2 \ll U_oT/L \ll 1$ . This is a limit of the preceding case; linearization is possible. Examples are : the initial stage of transient motion starting from rest, oscillating flow around a vibrating body, or wave motion (sound or sea waves) past a body (or a droplet, a bubble), etc.