Notes on
1.63 Advanced Environmental Fluid Mechanics
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December 1, 2002
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3.6 Karman's momentum integral approach

Ref: Schlichting: **Boundary layer theory**

With a general pressure gradient the boundary layer equations can be solved by a variety of modern numerical means. An alternative which can still be employed to simplify calculations is the momentum integral method of Karman. We explain this method for a transient boundary layer along the x-axis forced by an unsteady pressure gradient outside. This pressure gradient can be due to some unsteady and nonuniform flow such as waves or gust.

Let us limit our discussoin to a two dimentional flow. Recall the continuity

$$u_x + w_z = 0 \tag{3.6.1}$$

and boundary layer equation:

$$\rho(u_t + uu_x + wu_z) = -p_x + \mu u_{zz} \tag{3.6.2}$$

$$0 = -p_z \tag{3.6.3}$$

The boundary conditions are

$$u = w = 0, \qquad z = 0; \quad u \to U, \; w \to 0, \qquad z \to \infty. \tag{3.6.4}$$

For general $U(x,t)$ the longitudinal pressure gradient is

$$-p_x = \rho(U_t + UU_x) \tag{3.6.5}$$

so that

$$\rho(u_t + uu_x + wu_z) = \rho(U_t + UU_x) + \mu u_{zz} \tag{3.6.6}$$

Instead of solving the initial-boundary-value problem accurately we introduce a moment method by integrating the momentum equation across the boundary layer, then make a reasonable assumption on the velocity profile to get the longitudinal variation of the boundary layer thickness. This is called the Karman's momentum integral approximation in boundary layer theory.

Let us add the following two equations

$$-\mu u_{zz} = \rho \left\{ U_t + UU_x - (u_t + uu_x + wu_z) \right\}$$

and

$$0 = (U-u)u_x + (U-u)w_z$$

to get,

$$-\mu u_{zz} = \rho \left\{ \frac{\partial}{\partial t}(U-u) + \frac{\partial}{\partial x}[u(U-u)] + (U-u)\frac{\partial U}{\partial x} + \frac{\partial}{\partial z}[(w(U-u)] \right\}$$

Now let us integrate across the boundary layer,

$$\mu u_z|_0 = \rho \left\{ \frac{\partial}{\partial t}\int_0^\infty (U-u)dz + \frac{\partial}{\partial x}\int_0^\infty u(U-u)dz + \frac{\partial U}{\partial x}\int_0^\infty (U-u)dz \right\} + \rho[w(U-u)]_0^\infty$$

The boundary terms again vanish. Finally, we have,

$$\boxed{\frac{\partial}{\partial t}\int_0^\infty \rho(U-u)dz + \frac{\partial}{\partial x}\int_0^\infty \rho\left(Uu-u^2\right)dz + \frac{\partial U}{\partial x}\int_0^\infty (U-u)dz = \mu\left.\frac{\partial u}{\partial z}\right|_0} \tag{3.6.7}$$

This is the Kármán momentum integral equation, representing the momentum balance across the thickness of the boundary layer.

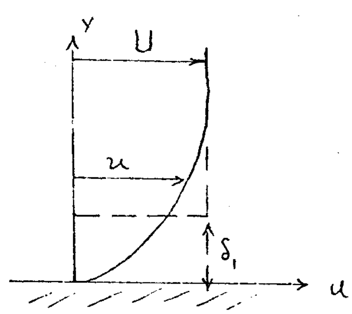

Figure 3.6.1: Displacement thickness

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Karman also introduces the displacement thickness as a measure of the lost volume

$$U\delta_1 = \int_0^\infty (U-u)\,dz, \quad \text{or} \quad \delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right)dz, \tag{3.6.8}$$

and the momentum thickness as a measure of the lost momentum

$$U^2\delta_2 = \int_0^\infty u(U-u)dz, \quad \text{or} \quad \delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dz, \tag{3.6.9}$$

both due to the slowing down of the fluid in the boundary layer. See Figure 3.6.1.

In certain cases when the velocity profile in the boundary layer can be reasonably guessed in advance, the Karman momentum intergal equation can be the basis of obtaining an approximate solution. The procedure is to assume a reasonable profile :

$$u = Uf\left(\frac{y}{\delta}\right) \tag{3.6.10}$$

and substitute (3.6.10) into (3.6.7) equation. After evaluating the integrals a differential equation is obtained for the boundary layer thickness $\delta(x,t)$, which can be more easily solved for certain initial and boundary conditions.