Notes on
1.63 Advanced Environmental Fluid Mechanics
Instructor: C. C. Mei, 2001
ccmei@mit.edu, 1 617 253 2994

December 1, 2002
3-2-bernoulli.tex

3.2 Bernoulli theorems of homogeneous fluids

3.2.1 Unsteady and irrotational flows

From the momentum equation,

$$\frac{\partial \vec{q}}{\partial t} + \nabla \frac{\vec{q}^2}{2} - \vec{q} \times (\nabla \times \vec{q}) = -\frac{1}{\rho} \nabla p + \vec{f}. \tag{3.2.1}$$

If the body force is conservative and the flow irrotational, i.e., $\vec{f} = -\nabla \Gamma$ and $\vec{q} = \nabla \phi$, then

$$\nabla \left[\frac{\partial \phi}{\partial t} + \frac{\vec{q}^2}{2} + \frac{p}{\rho} + \Gamma \right] = 0$$

which can be integrated in space to give

$$\frac{\partial \phi}{\partial t} + \frac{\vec{q}^2}{2} + \frac{p}{\rho} + \Gamma = C(t) \tag{3.2.2}$$

for all $\vec{x}$. This Bernoulli law is useful in the theory of surface waves.

3.2.2 Steady but rotational flows

The momentum equaton reads:

$$\vec{q} \cdot \nabla \vec{q} = -\frac{1}{\rho} \nabla p + \vec{f}$$

still applies. If $\rho =$ constant and $\vec{f} = -\nabla \Gamma$, we have

$$q_j \frac{\partial q_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \Gamma}{\partial x_i}$$

and, by scalar multiplication with q_i,

$$q_i \left(q_j \frac{\partial q_i}{\partial x_j} \right) = q_i \frac{\partial}{\partial x_i} \left(-\frac{p}{\rho} - \Gamma \right)$$

Now the left-hand side can be written as

$$q_j \frac{\partial}{\partial x_j} \frac{q_i q_j}{2} \quad \text{since} \quad \frac{\partial q_j}{\partial x_j} = 0$$

Therefore,

$$q_i \frac{\partial}{\partial x_i} \left[\frac{\vec{q}^2}{2} + \frac{p}{\rho} + \Gamma \right] = 0.$$

and

$$\frac{\vec{q}^2}{2} + \frac{p}{\rho} + \Gamma = \text{constant along a streamline.} \tag{3.2.1}$$

A streamline is a curve along which the velocity vectors are tangent to the line. It is importatn that the constant may be different for different streamlines, hence (3.2.1) is different from (3.2.2).