Notes on
1.63 Advanced Environmental Fluid Mechanics
Instructor: C. C. Mei, 2001
ccmei@mit.edu, 1 617 253 2994

December 1, 2002
3-10-oc-BL.tex

3.10 Oscillatory Boundary Layers

3.10.1 Stokes problem

If the viscosity is weak or the frequency is high (time scale is small) , there is a boundary layer near the boundary of an oscillating body or the solid bottom under a wave.

Let the external flow have the tangential velocity $u = \Re U(x)e^{-i\omega t}$. If the inertia parameter is small, then

$$\frac{uu_x, vu_y}{u_t} = O\left(\frac{U_o}{\omega L}\right) \ll 1.$$

If furthermore

$$1 \gg \frac{U}{\omega L} \gg \frac{\nu}{\omega L^2}$$

we keep keep just the most important viscous stress term and seek perturbation solution in successive powers of the inertia parameter. Let us save the trouble of nondimensionalization and assume

$$u = u_1 + u_2 + \dots \tag{3.10.1}$$

with the understanding that the relative orders of magnitude is represented by the subscripts, i.e., u_1 is the leading order and u_2 is smaller than u_1 by a factor $O(\frac{U}{\omega L} \gg \frac{\nu}{\omega L^2})$, etc. We get from the Navier-Stokes equations the leading order approximation

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial t} Re\left(Ue^{-i\omega t}\right) + \nu \frac{\partial^2 u_1}{\partial y^2} \qquad y > 0 \tag{3.10.2}$$

subject to the boundary conditions that

$$u_1 \to Re\, Ue^{-i\omega t} \qquad y \to \infty \tag{3.10.3}$$

and

$$u_1 = 0 \qquad y = 0 \tag{3.10.4}$$

Let

$$u_1 = Re\left[\hat{u}_1(x, y)e^{-i\omega t} + Ue^{-i\omega t}\right] \tag{3.10.5}$$

then

$$-i\omega U - i\omega \hat{u}_1 = -i\omega U + \nu \frac{d^2 \hat{u}_1}{dy^2}$$

Therefore,

$$\frac{d^2\hat{u}_1}{dy^2} + \frac{i\omega}{\nu}\hat{u}_1 = 0 \tag{3.10.6}$$

$$\hat{u}_1 \to 0, \qquad y \to \infty \tag{3.10.7}$$

$$\hat{u}_1 = -U_1(x), \qquad y = 0 \tag{3.10.8}$$

The solution is

$$\hat{u}_1 = -U(x) \exp\left[-(1-i)y\sqrt{\frac{\omega}{2\nu}}\right] \tag{3.10.9}$$

or,

$$u_1 = \Re\left\{U(x)\left[1 - \exp\left(-(1-i)y\sqrt{\frac{\omega}{2\nu}}\right)\right]e^{-i\omega t}\right\} \tag{3.10.10}$$

The sign of $\sqrt{-i}$ is chosen so that (3.10.7) is satisfied. The boundary layer thickness is

$$\delta = \sqrt{\frac{2\nu}{\omega}} \tag{3.10.11}$$

It is easy to show that the vorticity at the wall is

$$-\left.\frac{\partial u_1}{\partial y}\right|_{y=0} = (1-i)U\sqrt{\frac{\omega}{2\nu}} = e^{-i\pi/4}U \tag{3.10.12}$$

which has the phase lag of $\pi/4$ behind U, and diffuses away from $y = 0$ like a propagating wave exponentially attentuated after the Sokes boundary layer thickness $O(\delta)$.

3.10.2 Induced Streaming

By considering the small effect of convective inertia, the second order improvement is physically even more interesting.

If the inviscid outer flow has tangential variation $\frac{dU}{dx} \neq 0$, then by continuity there is transverse flow v_1 in the boundary layer :

$$\begin{aligned} v_1 &= -\int_0^y \frac{\partial u_1}{\partial x} dy = ie^{-i\omega t}\frac{dU}{dx}\int_0^y \left[1 - e^{-(1-i)y/\delta}\right] dy \\ &= -e^{-i\omega t}\frac{dU}{dx}\left\{y - \frac{\delta}{1-i}\left[1 - e^{-(1-i)y/\delta}\right]\right\} \end{aligned} \tag{3.10.13}$$

which is valid in $y \leq O(\delta)$ only.

Let us examine the second order:

$$\begin{aligned} \frac{\partial u_2}{\partial t} - \nu\frac{\partial^2 u_2}{\partial y^2} &= U\frac{dU}{dx} - \left(u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y}\right) \\ &= U\frac{dU}{dx} - \left[\frac{\partial(u_1 u_1)}{\partial x} + \frac{\partial(u_1 v_1)}{\partial y}\right] \end{aligned}$$

Taking average over a period

$$-\nu\frac{\partial^2\bar{u}_2}{\partial y^2} = \overline{U\,\frac{dU}{dx}} - \left(\frac{\partial\overline{u_1u_1}}{\partial x} + \frac{\partial\overline{u_1v_1}}{\partial y}\right)$$

On the right-hand-side the last two terms $\overline{u_1u_1}$, $\overline{u_1v_1}$ are wave-induced Reynolds stresses

$$\overline{u_1u_1} = \text{rate of transporting } x\text{-momentum in } x\text{-direction}$$

$$\overline{u_1v_1} = \text{rate of transporting } x\text{-momentum in } y\text{-direction}$$

Alternatively:

$$-\nu\frac{\partial^2\bar{u}_2}{\partial y^2} = \frac{1}{2}\frac{\partial}{\partial x}\overline{U^2} - \frac{1}{2}\frac{\partial}{\partial x}\overline{u_1^2} - \overline{v_1\frac{\partial u_1}{\partial y}}$$

Let

$$\alpha = (1-i)/\delta \tag{3.10.14}$$

Since

$$v_1 = ie^{-i\omega t}\frac{1}{\alpha}\frac{dU}{dx}\left(\alpha y - 1 + e^{-\alpha y}\right)$$

$$\frac{\partial u_1}{\partial y} = \alpha U(x)e^{-i\omega t}e^{-\alpha y}$$

$$\overline{-v_1\,\frac{\partial u_1}{\partial y}} = \frac{1}{2}Re\left[U^*\frac{dU}{dx}\frac{\alpha^*}{\alpha}e^{-\alpha^* y}\left(\alpha y - 1 + e^{-\alpha y}\right)\right]$$

Thus

$$\begin{aligned}
-\nu\frac{\partial^2\bar{u}_2}{\partial y^2} &= G(y) \equiv \frac{1}{2}\frac{d|U|^2}{dx}\left[1 - \left(1 - e^{-\alpha y}\right)\left(1 - e^{-\alpha^* y}\right)\right] \\
&\quad + Re\,U^*\frac{dU}{dx}\frac{\alpha *}{\alpha}e^{-\alpha^* y}\left(\alpha y - 1 + e^{-\alpha y}\right) \\
\nu\,\frac{\partial\bar{u}_2}{\partial y} &= \int_y^\infty G(y')dy' \\
\nu\,\bar{u}_2 &= \int_0^y dy''\int_{y''}^\infty G(y')dy' \\
&= -y\int_y^\infty G(y')dy' + \int_0^y y''G(y'')dy''
\end{aligned}$$

One more integration gives

$$\begin{aligned}
-\omega\bar{u}_2 &= Re\,F_4\,U\frac{dU^*}{dx}. \quad \text{where} \\
F_4 &= -\frac{1}{2}(1-3i)e^{-(1-i)\eta} - \frac{i}{2}e^{-(1+i)\eta} - \frac{1}{4}(1+i)e^{-2\eta} \\
&\quad + \frac{1}{2}(1+i)\eta e^{-(1-i)\eta} + \frac{3}{4}(1-i)
\end{aligned}$$

Note that as $y \to \infty$, just outside the boundary layer,

$$\bar{u}_2 = -\frac{1}{4\omega} Re\left[(3-i)\, U\frac{dU^*}{dx}\right]$$

Let $U = Ae^{i\gamma}$

$$U\frac{dU^*}{dx} = Ae^{i\gamma}\frac{dAe^{-i\gamma}}{dx} = \frac{dA^2/2}{dx} - iA^2\frac{d\gamma}{dx}$$

Hence

$$\bar{u}_2(\infty) = -\frac{1}{4\omega}\left(\frac{3}{2}\frac{dA^2}{dx} - 3A^2\frac{d\gamma}{dx}\right) \tag{3.10.15}$$

Example: Progressive waves, $U = U_o e^{ikx}$, where U_o, k are constants

$$\bar{u}_2(\infty) = \frac{3k}{4\omega}U_o^2 \tag{3.10.16}$$

3.10.3 Physics of the Induced Streaming

Take a progressive water waves as an example:

Outside the B.L. :

$$u_\infty = A\,\cos(\omega t - kx) \tag{3.10.17}$$

Inside the B.L.

$$u = A\left[\cos(\omega t - kx) - e^{-y/\delta}\cos(\omega t - kx - y/\delta)\right] \tag{3.10.18}$$

where the velocity amplitude A is related to the surface amplitude "a" by

$$A = a\omega/\sin h\, kh \tag{3.10.19}$$

Let's find the induced transverse velocity v

$$\frac{\partial u}{\partial x} = A\,\sin(\omega t - kx) - Ae^{-y/\delta}\,\sin(\omega t - kx - y/\delta)$$

$$v_\infty = -\int_0^{y\gg\delta}\frac{\partial u}{\partial x}\,dy = -y\,A\,\sin(\omega t - kx) - \frac{1}{2}Ak\delta\,\cos(\omega t - kx) + \frac{1}{2}Ak\delta\,\sin(\omega t - kx)$$

Now

$$\overline{u_\infty v_\infty} = -\frac{1}{4}A^2k\delta < 0$$

where the $\sin(\omega t - kx)$ terms in v_∞ are out of phase with u_∞ by $\pi/2$, hence does not contribute to the mean.

Now consider a slice of boundary layer one wavelength long. Because of periodicity, there is no net transfer of momentum or forces at two ends x_0 and $x_0 + 2\pi/k$. But the momentum transfer downwards is $\frac{A^2}{4}k\delta$, causing a positive shear stress. To balance it there must be a non-zero $\mu\frac{\partial\bar{u}}{\partial y}$ at all levels y below the top. Hence, $\bar{u} \neq 0$, resulting in induced streaming.

Homework: Find the induced streaming in Stokes boundary layer under a standing wave with

$$U(x) = A\cos kx\, e^{-i\omega t}.$$

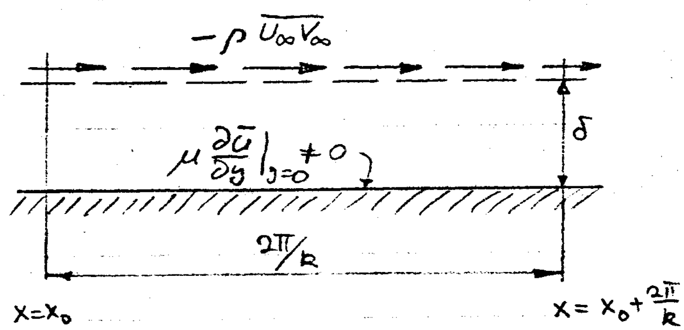

Figure 3.10.1: Reynolds stress and Induced streaming in Stokes layer