Notes on
1.63 Advanced Environmental Fluid Mechanics
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CHAPTER 3
High Speed flows

In this chapter we examine high-speed flows of a viscous fluid. As a preparation, the limit of inviscid flows is breifly discussed.

3.1 Irrotational flows of homogeneous fluids

For an inviscid and incompressible fluid with constant density,

$$\frac{D\vec{\zeta}}{Dt} = \vec{\zeta} \cdot \nabla \vec{q}.$$

If $\vec{\zeta} = 0$ everywhere at $t = t_0$, then

$$\frac{D\vec{\zeta}}{Dt} = 0$$

at $t = t_0$ for all $\vec{x}$. Therefore, at $t = t_0 + dt'$, $\vec{\zeta} = 0$ everywhere. Repeating the argument, $\vec{\zeta}$ remains zero at $t = t_0 + 2dt, t_0 + 3dt, \ldots$, for all $\vec{x}$. In other words the flow is irrotational at all times if it is so at the start. A flow in which $\vec{\zeta} = \nabla \times \vec{q}$ vanishes everywhere is called an irrotational flow.

It is a well known identity in vector analysis that an irrotational vector can be expressed as the gradient of a scalar potential. Thus we define the velocity potential ϕ by

$$\vec{q} = \nabla \phi \tag{3.1.1}$$

An immediate consequence of continuity is that

$$\nabla \cdot \vec{q} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \tag{3.1.2}$$

i.e., ϕ is a harmonic function of $\vec{x}$. Note that (3.1.2) is the result of mass and momentum conservation.

3.1.1 Two-dimensional Irrotational Flows

If the motion is two-dimensional in the x, y plane then continuity equation reads:

$$\nabla \cdot \vec{q} = 0 \qquad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1.3}$$

and irrotationality requires

$$\nabla \times \vec{q} = 0 \qquad \text{or} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{3.1.4}$$

3.1.2 Velocity potential

In the mathematical theory of complex variables, two functions of x, y that satisfy these two *Cauchy-Riemann conditions* are called are harmonic congugates of each other. As noted in the Because of irrrotationality, (3.1.4), the velocity components can be expressed as the gradient of a potential ϕ

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \tag{3.1.5}$$

so as to satisfy (3.1.4) automatically. Substitute these into the law of mass conservation, (3.1.3), we find ϕ to be a harmonic function

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{3.1.6}$$

3.1.3 Stream function

From two dimensional problems it is useful to introduce another scalar function, the stream function ψ, to satisfy (3.1.3), automatically,

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}. \tag{3.1.7}$$

Substituting Eqn. (3.1.7) into Eqn. (3.1.4), we find ψ to be a harmonic function too.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{3.1.8}$$

By definition,

$$(u =)\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad (v =)\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{3.1.9}$$

therefore ϕ and ψ also satisfy Cauchy-Riemann conditons and are harmonic conjugates of each other. This is why the theory of complex functions is so important is two dimensional potential flows.

In the plane of x, y, lines of constant ϕ are called equipotential lines; the velocity vector is normal to equipotential lines and is directed from lower to higher poteitials. Lines of constant ψ are the streamlines; the velocity vector is tangential to the local streamline. It follows that equi-potentials are perpendicular to streamlines. As a formal proof we note that

$$\nabla\phi \cdot \nabla\psi = \phi_x\psi_x + \phi_y\psi_y = u(-v) + vu = 0 \tag{3.1.10}$$

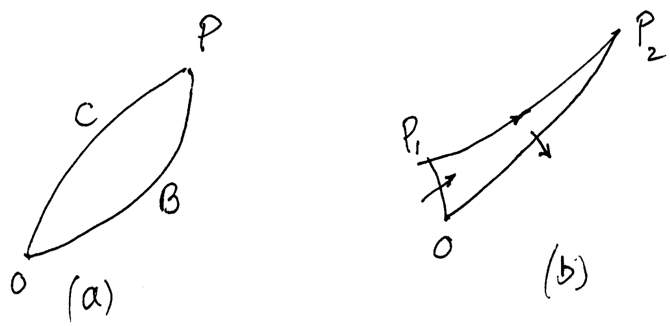

Figure 3.1.1: Definition of the stream function

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Indeed the difference of the stream functions at two points is just the volume flux rate between the two points. This can be seen by using the definitions (3.1.7). First $\psi(x, y)$ has the dimension of volume flux rate : $UL = L^2/T$. With reference to Figure (**??**), the flux between two streamlines can be calculated in two equivalent ways

$$u\delta y \,(\text{along } x = \text{ constant}) = -v\delta x \,(\text{along } y = \text{ constant})$$

In view of (3.1.7),

$$u = \frac{\partial\psi}{\partial y} = \frac{\delta\psi}{\delta y}\,|_{x=\text{const.}}, \quad v = -\frac{\partial\psi}{\partial x} = -\frac{\delta\psi}{\partial x}\,|_{y=\text{const.}},$$

hence

$$u\delta y = \frac{\delta\psi}{\partial y}\delta y = \delta\psi, \qquad -v\delta x = \frac{\delta\psi}{\partial x}\delta x = \delta\psi.$$

where $\delta\psi = \psi_2 - \psi_1$. simple observations will confirm that the stream funciton has all the features of the rate of volume flux. From the theory of complex functions, the following complex potential

$$w = \phi(x, y) + i\psi(x, y) \qquad i = \sqrt{-1}. \tag{3.1.11}$$

is analytic in $z = x + iy$, except at singular points. In particular the derivative is independent of direction. Indeed,

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{i\partial y}$$

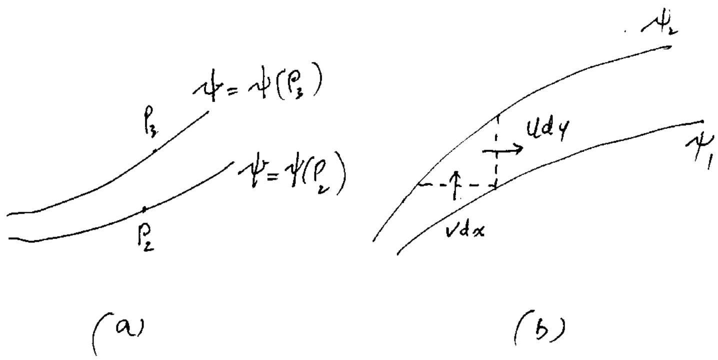

Figure 3.1.2: Physical meaning of the stream function

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since

$$\frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv, \quad \text{and} \quad \frac{\partial w}{i\partial y} = -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y} = u - iv.$$

Because of these connections to complex variables, the theory of analytical functions has been a powerful tool for solving 2D irrotational flow problems for a long time. Its luster has faded somewhat only after the advent of computers.