Notes on
1.63 Advanced Environmental Fluid Mechanics
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4-5disp-sh.tex. **Abbreviated version for steady flow only .**

Refs:

1. Fischer, List, Koh. Imberger and Brooks : *Mixing in Inland and Coastal Waters.*
2. Brenner & Edwards : *Macrotransport Processes* . Butterworth-Heinemann, 1993.

Macro

4.5 Dispersion of supension in a steady shear flow

Similar to heat, a passive solvent is also transported by convection and diffusion. When the concentration of the solvent is low, the flow affects the transport but the solvent does not affect the flow. We explain a systematic theory of enhanced diffusion by shear- a phenomenon first analysized by G.I. Taylor, hence is called the Taylor dispersion.

4.5.1 The convective diffusion equation for a passive suspension

The spreading of a passive solute in a flow is governed by the convective diffusion equation

$$\frac{\partial C}{\partial t} + \nabla \cdot (\mathbf{q}C) = D\nabla^2 C \tag{4.5.1}$$

where C is the volume concentration of the solute, and D the diffusivity. The derivation is as follows.

Consider a material volume V. Conservation of the solute mass requires:

$$\frac{D}{Dt}\iint_V C dV = -\iint_S \mathbf{f}\cdot\mathbf{n} dS = -\iint_V \nabla\cdot\mathbf{f} dV \tag{4.5.2}$$

The left hand side can be written as

$$\iint_V \left(\frac{\partial C}{\partial t} + \nabla\cdot(\mathbf{q}C)\right) dV \tag{4.5.3}$$

Assuming Fick's law

$$\mathbf{f} = -D\nabla C \tag{4.5.4}$$

we get

$$\iint_V dV \left(\frac{\partial C}{\partial t} + \nabla\cdot(\mathbf{q}C)\right) = \iint_V dV\, D\nabla^2 C \tag{4.5.5}$$

Since V is arbitrary, (4.5.1) follows.

4.5.2 Heuristic picture of shear-enhanced diffusion and scales estimates

If a streak of die is put initially straight aross a channel of fluid in steady uniform flow, after some time the die streak will become curved due to the transverse variation of velocity, and also becomes thicker due to diffusion in all directions. See figure 4.5.1. After sufficiently long time lateral diffusion will be complete and the die distribution is nearly uniform across the channel. Convective diffusion will continue as a one-dimensional process. We wish to predict the effecitive dissfusivity in the last phase. Consider dispersion in a one-dimensional shear

4.1 Dispersion in Laminar Shear Flow 87

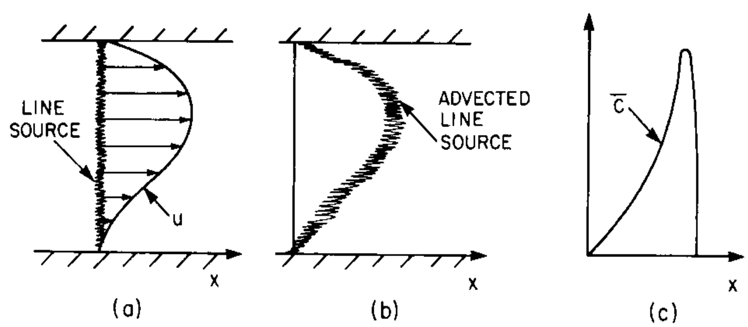

Figure 4.4 The production of a skewed longitudinal distribution by advection of a line source. (a) The line source and the velocity profile. (b) The advected line source. (c) The longitudinal distribution of the cross-sectional mean concentration corresponding to the distribution shown in (b).

Figure 4.5.1: Initial stage of convective diffusion of a die streak. From Fischer et al.

flow in a long circular pipe resulting from a pressure gradient which is constant in space but has steady and oscillatory parts. The velocity profile is known,

$$u = U_s(r), \quad 0 < r < a. \tag{4.5.6}$$

The concentration of a solvent is governed by

$$\frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} = D \left(\frac{\partial^2 C}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial C}{\partial r} \right) \right) \tag{4.5.7}$$

There are three time scales : diffusion time across the pipe radius a, convection time across L, and diffusion time across L. Their ratios are :

$$\frac{a^2}{D} : \frac{L}{U_o} : \frac{L^2}{D} = 1 : \frac{1}{\epsilon} : \frac{1}{\epsilon^2} \tag{4.5.8}$$

The first time scale for lateral diffusion is very small. Let us focus attention to processes in the longitudinal direction long after lateral diffusion is complete. Hence we choose the following normalization,

$$x = Lx',\, r = ar',\, u = U_o u',\, t = \frac{L}{U_o} t', \tag{4.5.9}$$

where U_o is the scale of U. Equation (4.5.7) is normalized to

$$\frac{U_o a}{D}\frac{a}{L}\left(\frac{\partial C'}{\partial t'} + \frac{\partial (u'C')}{\partial x'}\right) = \frac{a^2}{L^2}\frac{\partial^2 C'}{\partial x'^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) \tag{4.5.10}$$

Let the Péclét number $Pe = U_o a/D$ be of order unity and $a/L = \epsilon \ll 1$. Equation (4.5.10) becomes

$$\epsilon Pe\left(\frac{\partial C'}{\partial t'} + \frac{\partial (u'C')}{\partial x'}\right) = \epsilon^2\frac{\partial^2 C'}{\partial x'^2} + \frac{1}{r}\frac{\partial}{\partial r'}\left(r'\frac{\partial C'}{\partial r'}\right) \tag{4.5.11}$$

with the boundary conditons

$$\frac{\partial C'}{\partial r'} = 0, \quad r' = 0, 1 \tag{4.5.12}$$

with

$$u' = U'_s \tag{4.5.13}$$

For brevity we drop the primes from now on.

4.5.3 Multiple scale analysis-homogenization

Ignoring the early stage of lateral difrusion, there are two time scales : convection time across L, and diffusion time across L. Their ratios are :

$$\frac{L}{U_o} : \frac{L^2}{D} = 1 : \frac{1}{\epsilon} \tag{4.5.14}$$

Therfore we introduce the multiple time coordinates

$$t,\, t_1 = \epsilon t, \tag{4.5.15}$$

and consider $C(x, r, t, t_1)$ to depend on t, and t_1 as if they were independent. In particular,

$$\frac{\partial C}{\partial t} \to \frac{\partial C}{\partial t} + \frac{\partial C}{\partial t_1}\frac{\partial t_1}{\partial t} = \frac{\partial C}{\partial t} + \epsilon\frac{\partial C}{\partial t_1} \tag{4.5.16}$$

Introduce multiple scale expansions.

$$C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \dots. \tag{4.5.17}$$

with $C_i = C_i(x, r, t, t_1)$, for all $i = 1, 2, 3 \cdots$. The perturbation problems are

$O(\epsilon^0)$:

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_0}{\partial r}\right) \tag{4.5.18}$$

with the boundary conditions:

$$\frac{\partial C_0}{\partial r} = 0, \quad r = 0, 1. \tag{4.5.19}$$

$O(\epsilon)$:

$$Pe\left(\frac{\partial C_0}{\partial t} + \frac{\partial (uC_0)}{\partial x}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_1}{\partial r}\right) \tag{4.5.20}$$

with:

$$\frac{\partial C_1}{\partial r} = 0, \quad r = 0, 1. \tag{4.5.21}$$

$O(\epsilon^2)$:

$$Pe\left(\frac{\partial C_0}{\partial t_1} + \frac{\partial C_1}{\partial t} + \frac{\partial (uC_1)}{\partial x}\right) = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_2}{\partial r}\right) \tag{4.5.22}$$

with

$$\frac{\partial C_2}{\partial r} = 0, \quad r = 0, 1. \tag{4.5.23}$$

The solution at $O(\epsilon^0)$ is

$$C_0 = C_0(x, t_1), \tag{4.5.24}$$

At $O(\epsilon)$, let the known velocity be

$$u = U_s(r) \tag{4.5.25}$$

we get

$$Pe\left(\frac{\partial C_0}{\partial t} + U_s\frac{\partial C_0}{\partial x}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_1}{\partial r}\right) \tag{4.5.26}$$

with

$$\frac{\partial C_1}{\partial r} = 0, \quad r = 0, 1 \tag{4.5.27}$$

Let us now integrate(or average) (4.5.26)across the pipe, and get

$$\frac{\partial C_0}{\partial t} + \langle U_s\rangle\frac{\partial C_0}{\partial x} = 0 \tag{4.5.28}$$

where angle brackets denote averaging over the cross section.

$$\langle h\rangle = \frac{1}{\pi}\int_0^1 2\pi r h\, dr$$

Now subtract Pe(4.5.28) from (4.5.26)

$$Pe\tilde{U}_s\frac{\partial C_0}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_1}{\partial r}\right) \tag{4.5.29}$$

where

$$\tilde{U} = U_s(r) - \langle U_s \rangle \tag{4.5.30}$$

is the velocity nonuniformity

Now C_1 is governed by a linear equation, we can assume the solution to be proportional to the forcing, i.e.,

$$C_1 = Pe\frac{\partial C_0}{\partial x}B_s(r) \tag{4.5.31}$$

then

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_s}{dr}\right) = \tilde{U}(r) \tag{4.5.32}$$

with the boundary conditions

$$\frac{dB_s}{dr} = 0, \quad r = 0, 1. \tag{4.5.33}$$

and

After solving for B_s we go to $O(\epsilon^2)$, i.e., (4.5.22) . Note that , in view of (4.5.31),

$$Pe\frac{\partial uC_1}{\partial x} = Pe^2(\langle U_s \rangle + \tilde{U}_s)B_s(r)\frac{\partial^2 C_0}{\partial x^2}$$

hence,

$$Pe\left(\frac{\partial C_0}{\partial t_1} + \frac{\partial C_1}{\partial t_1}\right) + Pe^2(\langle U_s \rangle + \tilde{U}_s)B_s(r)\frac{\partial^2 C_0}{\partial x^2} = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_2}{\partial r}\right) \tag{4.5.34}$$

which is a linear PDE for C_2. From (4.5.31) and (4.5.28) we find

$$Pe\frac{\partial C_1}{\partial t} = -Pe^2\frac{\partial^2 C_0}{\partial x^2}\langle U_s \rangle B_s(r) \tag{4.5.35}$$

It follows that

$$Pe\frac{\partial C_0}{\partial t_1} + Pe^2\tilde{U}_s B_s\frac{\partial^2 C_0}{\partial x^2} = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C_2}{\partial r}\right) \tag{4.5.36}$$

with

$$\frac{\partial C_2}{\partial r} = 0 \quad r = 0, 1 \tag{4.5.37}$$

Averaging (4.5.36) across the pipe, we get

$$\frac{\partial C_0}{\partial t_1} = E\frac{\partial^2 C_0}{\partial x^2} \tag{4.5.38}$$

with

$$E = 1 - Pe^2 \langle \tilde{U}_s B_s \rangle \tag{4.5.39}$$

which is the effective diffusion coefficient. The first part is of molecular origin; the second part (the dispersion coefficient) is due to fluid shear.

Finally we add ϵ (4.5.28) and (4.5.38) to get:

$$Pe \left(\frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial t_1} \right) C_0 + Pe \langle U_s \rangle \frac{\partial C_0}{\partial x} = \epsilon E \frac{\partial^2 C_0}{\partial x^2} \tag{4.5.40}$$

This describes the convective diffusion of the area averaged concentration, which is certainly of practical value.

After the perturbation analysis is complete, there is no need to use multiple scales; we may now write

$$\frac{\partial C_0}{\partial t} + Pe \langle U_s \rangle \frac{\partial C_0}{\partial x} = \epsilon E \frac{\partial^2 C_0}{\partial x^2} \tag{4.5.41}$$

In dimensionless form, it reads,

$$\frac{\partial C_0}{\partial t} + \langle U_s \rangle \frac{\partial C_0}{\partial x} = DE \frac{\partial^2 C_0}{\partial x^2} \tag{4.5.42}$$

This equation governs the convective diffusion of the cross-esctional average, after the initial transient is smoothed out.

4.5.4 The dispersion coefficient for steady flow

We demonstrate the details for a pure steady flow.

The dimensional velocity profile is

$$U_s(r) = U_o \left(1 - \frac{r^2}{a^2} \right) \tag{4.5.43}$$

with U_o begin the maximum at the centerline. It is easy to show that

$$\langle U \rangle = \frac{U_o}{2} \tag{4.5.44}$$

We shall use U_o as the velocity scale for normalization, then in dimensionless form

$$u(r) = 1 - r^2, \quad \langle u \rangle = \frac{1}{2} \tag{4.5.45}$$

and the nonuniform part is

$$\tilde{u} = \frac{1}{2} - r^2 \tag{4.5.46}$$

The equation for B_s is

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dB_s}{dr} \right) = \frac{1}{2} - r^2 \tag{4.5.47}$$

which can be easily integrated to give

$$B_s = \frac{1}{8}\left(r^2 - \frac{r^4}{2}\right) + B' \tag{4.5.48}$$

To detemine the integration constant we impose the condition that

$$\langle B_s \rangle = 2\int_0^1 rB_s dr = 0 \tag{4.5.49}$$

This implies

$$0 = B' + \frac{1}{24} \tag{4.5.50}$$

hence

$$B_s = -\frac{1}{24} + \frac{1}{8}\left(r^2 - \frac{r^4}{4}\right) \tag{4.5.51}$$

Using (4.5.46) and (4.5.51), we get

$$\langle \tilde{u} B_s \rangle = 2\int_0^1 r\left(\frac{1}{2} - r^2\right)\left[-\frac{1}{24} + \frac{1}{8}\left(r^2 - \frac{r^4}{4}\right)\right] dr = -\frac{1}{192} \tag{4.5.52}$$

Thus

$$\frac{\partial C_0}{\partial t_1} + Pe\langle U_s \rangle \frac{\partial C_0}{\partial x} = \epsilon\left(1 + \frac{Pe^2}{192}\right)\frac{\partial^2 C_0}{\partial x^2} \tag{4.5.53}$$

In physical variables

$$\frac{\partial C_0}{\partial t} + \langle U_s \rangle \frac{\partial C_0}{\partial x} = D\left[1 + \frac{1}{192}\left(\frac{U_o a}{D}\right)^2\right]\frac{\partial^2 C_0}{\partial x^2} \tag{4.5.54}$$

Denote by K the shear-enhanced dispersion coifficient,

$$K = \frac{1}{192}\frac{(U_o a)^2}{D} \tag{4.5.55}$$

If the Peclet number is not small, the enhanced diffusivity (dispersivity) can be greater than the molecular diffusivity. Since the dispersivity is inversely proportional to D, its importance becomes greater for smaller molecular diffusisivity.

Example: For salt in water, $D = 10^{-5} cm^2/s$. If $U_o = 1cm/s$, and $a = 0.2cm$, then $K = 22cm^2/s$, far exceeding the molecular diffusivity.

Homework: Show that the shear-enhanced dispersion coefficient E for the uniform flow in a wide and inclined channel of depth h is $(8/945)U_o^2 h^2/D$.

Homework: Find the dispersion coefficient E in the oscillatory flow in a circular pipe.

Homework: Find the dispersion coefficient E in the oscillatory flow in a blood vessel with elastic wall.

Homework: Find the dispersion coefficient for sediments in an open channel flow. The sediments are uniform with fall velocity W_o.