Notes on
1.63 Advanced Environmental Fluid Mechanics
Instructor: C. C. Mei, 2002
ccmei@mit.edu, 1 617 253 2994

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4-3MTwind.tex

4.3 Convection driven by buoyancy - Mountain Wind

ref: Prandtl: **Fluid Dynamics.**

Due to solar heating during the day, a mountain slope may be warmer than the surrounding air in a summer night. Let the air near a mountain slope be stably stratified

$$T_o = T_0 + Ny', \tag{4.3.1}$$

where T_0 = constant, and $N > 0$. Let the slope temperature be :

$$T_s = T_1 + Ny', \tag{4.3.2}$$

where $T_1 > T_0$. See the left of Figure 4.3.2. Consider first the static equilibrium:

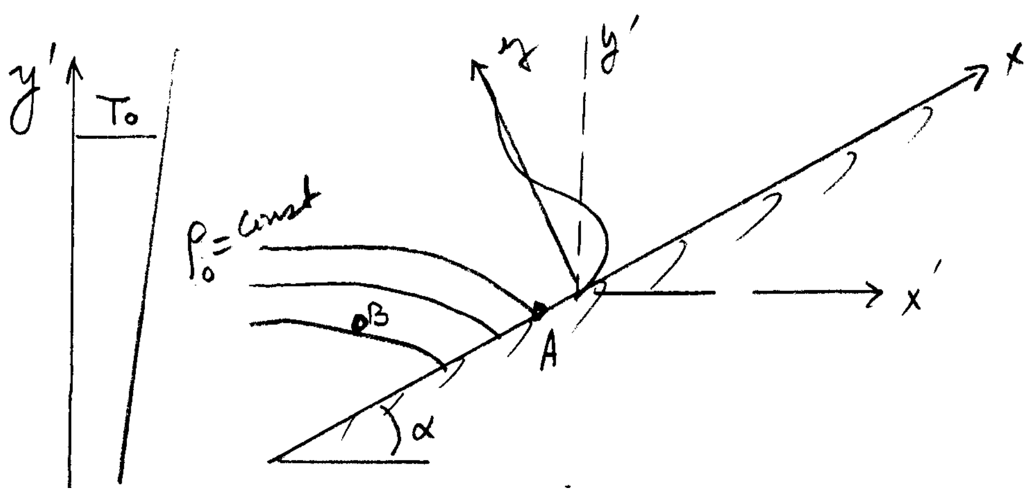

Figure 4.3.1: Thermal convection along a slope

$$0 = -\frac{dp_o}{dz} - \rho_o g$$

hence

$$p_o = p_o(\infty) + \int_z^\infty \rho_o \, g \, dz$$

Let A and B be two points with the same elevation but A is on the slope and B is in the air. Since $p_A < p_B$,

$$\frac{\partial p_o}{\partial x} < 0$$

implying

$$\frac{\partial p_o}{\partial x'} < 0$$

The pressure gradient must drive an upward flow along the slope.

Let us consider the dynamics. Let

$$T(x, y) = T_o + \theta(y) \tag{4.3.3}$$

and

$$\rho(x, y) = \rho_o + S(y) = \text{static density } + \text{dynamic density} \tag{4.3.4}$$

By the equation of state,

$$\rho = \rho_0 \left[1 - \beta\left(T - T_0\right)\right] = \rho_0 \left[1 - \beta\left(T_o - T_0\right)\right] - \rho_0 \beta \theta.$$

Therefore

$$\rho_o = \rho_0 \left[1 - \beta\left(T_o - T_0\right)\right] = \rho_0 \left(1 - \beta N y'\right) \tag{4.3.5}$$

and

$$S(x, y) = -\rho_0 \beta \theta(x, y) \tag{4.3.6}$$

Note by ratation of coordinates,

$$T_o - T_0 = N y' = N(x \sin\alpha + y \cos\alpha). \tag{4.3.7}$$

The flow equations are:

$$u_x + v_y = 0 \tag{4.3.8}$$

$$\rho\left(u u_x + v u_y\right) = -p_{dx} + \mu\left(u_{xx} + u_{yy}\right) - \left(\rho - \rho_a\right) g \sin\alpha \tag{4.3.9}$$

$$\rho\left(u v_x + v v_u\right) = -p_{dy} + \mu\left(v_{xx} + v_{yy}\right) - \left(\rho - \rho_a\right) g \cos\alpha \tag{4.3.10}$$

$$u T_x + v T_y = k\left(T_{xx} + T_{yy}\right), \tag{4.3.11}$$

where T is the total temperature and

$$k = \frac{K}{\bar{\rho}_o c_p}.$$

is the thermal diffusivity. Since $\partial/\partial x = 0$, $v = 0$ from continuity. From Eqn. (4.3.9)

$$\nu u_{yy} + \left(\beta g \sin\alpha\right)\theta = 0. \tag{4.3.12}$$

after invoking Boussinesq approximation. In Eqn. (4.3.11),

$$\frac{\partial T}{\partial x} = \frac{\partial T_o}{\partial x} = N \sin \alpha.$$

Therefore,

$$uN \sin \alpha = k\theta_{yy}. \tag{4.3.13}$$

Combining Eqns. (4.3.12) and (4.3.13), we get

$$\frac{d^4 u}{dy^4} + \left(\frac{\beta g N \sin^2 \alpha}{\nu k} \right) u = 0 \tag{4.3.14}$$

and

$$\frac{d^4 \theta}{dy^4} + \left(\frac{\beta g N \sin^2 \alpha}{\nu k} \right) \theta = 0 \tag{4.3.15}$$

Let

$$\ell^4 = \frac{4\nu k}{\beta g A \sin^2 \alpha} \text{ and } \quad y = \ell \eta \tag{4.3.16}$$

then

$$\frac{d^4 u}{d\eta^4} + 4u = 0; \quad \text{and} \quad \frac{d^4 \theta}{d\eta^4} + 4\theta = 0 \tag{4.3.17}$$

The velocity is

$$u = U\, e^{-\eta} \sin \eta \quad \text{so that} \quad u(0) = 0 \tag{4.3.18}$$

The temperature is

$$\theta = \theta_0 e^{-\eta} \cos \eta \tag{4.3.19}$$

The boundary conditions at $\eta \sim \infty$ are satisfied. In order that $\theta(0) = T_1 - T_0$ on $\eta = 0$ we choose

$$\theta_0 = T_1 - T_0 \tag{4.3.20}$$

Note that the boundary layer thickness is

$$\delta \sim O(\ell) \sim \left(\frac{4\nu k}{\rho g N \sin^2 \alpha} \right)^{1/4} \tag{4.3.21}$$

Thus if $\alpha \downarrow, \delta \uparrow$ as $1/\sin^2 \alpha$.

Using Eqn. (4.3.13), we get

$$N \sin \alpha\, U\, e^{-\eta} \, \sin \eta = k \left(\frac{\beta g N \sin^2 \alpha}{4\nu k} \right)^{1/2} 2\theta_0\, e^{-\eta} \sin \eta.$$

Hence,

$$U = \theta_0 \left(\frac{\beta g k}{N \nu} \right)^{1/2} \tag{4.3.22}$$

Finally

$$u = (T_1 - T_0)\left(\frac{\beta g k}{N\nu}\right)^{1/2} e^{-\eta}\sin\eta. \tag{4.3.23}$$

and

$$\theta = (T_1 - T_0)e^{-\eta}\cos\eta \tag{4.3.24}$$

It is easy to show from (4.3.13) that the total mass flux rate is

$$M = \int_0^\infty \rho\, u\, dy = \rho_0\, \beta\, k\, \left.\frac{d\theta}{dy}\right|_0 . \tag{4.3.25}$$

Note from (4.3.22) that U is independent of α. If $\alpha \downarrow$, the buoyancy force is weaker, but the shear rate $\partial u/\partial y$ is smaller, hence the wall resistance is smaller. U is not reduced!

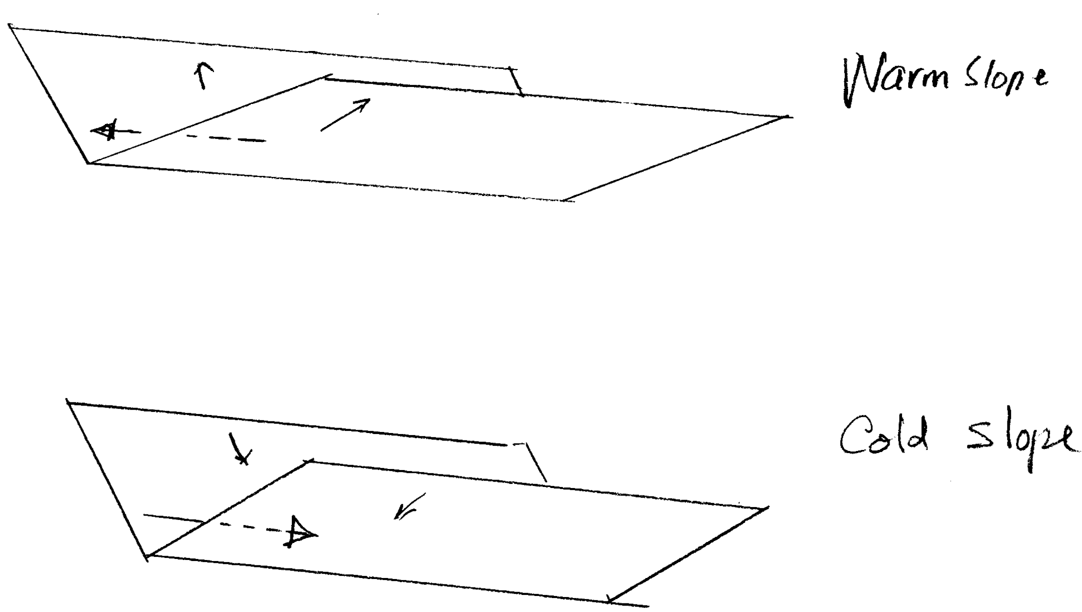

Figure 4.3.2: Wind along a valley due to feeding from mountains

On a warm slope (due to solar heating during the day) , air rises at night. If there are two slopes forming a valley, fluid must be supplied from the bottom of the valley; this is the reason for valley wind blowing from low altitude to high.

On a cold slope (due to radiation loss at night) air sinks at high noon. Valley wind must flow from high to low.