

Notes on
1.63 Advanced Environmental Fluid Mechanics
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Chapter 4. Thermal effects on fluid flow

4.1 Heat and energy conservation

Recall the basic equations for a compressible fluid. Mass conservation requires that :

$$\rho_t + \nabla \cdot \rho \vec{q} = 0 \quad (4.1.1)$$

Momentum conservation requires that :

$$\rho (\vec{q}_t + \vec{q} \nabla \cdot \vec{q}) = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \rho \vec{f} \quad (4.1.2)$$

where the viscous stress tensor $\bar{\bar{\tau}}$ has the components

$$\left(\bar{\bar{\tau}}\right)_{ij} = \tau_{ij} = \mu \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} \right) + \lambda \frac{\partial q_k}{\partial x_k} \delta_{ij}$$

There are 5 unknowns ρ , p , q_i but only 4 equations. One more equation is needed.

4.1.1 Conservation of total energy

Consider both mechanical and thermal energy. Let e be the internal (thermal) energy per unit mass due to microscopic motion, and $q^2/2$ be the kinetic energy per unit mass due to macroscopic motion. Conservation of energy requires

$$\begin{aligned} \frac{D}{Dt} \iiint_V \rho \left(e + \frac{q^2}{2} \right) dV & \quad \text{rate of incr. of energy in } V(t) \\ & = - \iint_S \vec{Q} \cdot \vec{n} dv \quad \text{rate of heat flux into } V \\ & + \iiint_V \rho \vec{f} \cdot \vec{q} dV \quad \text{rate of work by body force} \\ & + \iint_X \vec{\Sigma} \cdot \vec{q} dS \quad \text{rate of work by surface force} \end{aligned}$$

Use the kinematic transport theorem, the left hand side becomes

$$\iiint_V \rho \frac{D}{Dt} \left(e + \frac{q^2}{2} \right) dV$$

Using Gauss theorem the heat flux term becomes

$$- \iint_S Q_i n_i dS = - \iiint_V \frac{\partial Q_i}{\partial x_i} dV$$

The work done by surface stress becomes

$$\begin{aligned} \iint_S \Sigma_j q_j dS &= \iint_S (\sigma_{ji} n_i) q_j dS \\ &= \iint_S (\sigma_{ij} q_j) n_i dS = \iiint_V \frac{\partial (\sigma_{ij} q_j)}{\partial x_i} dV \end{aligned}$$

Now all terms are expressed as volume integrals over an **arbitrary** material volume, the following must be true at every point in space,

$$\rho \frac{D}{Dt} \left(e + \frac{q^2}{2} \right) = - \frac{\partial Q_i}{\partial x_i} + \rho f_i q_i + \frac{\partial (\sigma_{ij} q_j)}{\partial x_j} \quad (4.1.3)$$

As an alternative form, we differentiate the kinetic energy and get

$$\rho \frac{De}{Dt} + \underbrace{\rho q_i \frac{Dq_i}{Dt}} = \underbrace{\rho f_i q_i + q_i \frac{\partial \sigma_{ij}}{\partial x_j}} - \frac{\partial Q_i}{\partial x_i} - p \frac{\partial q_i}{\partial x_i} + \frac{\partial (\tau_{ij} q_i)}{\partial x_j} \quad (4.1.4)$$

Because of momentum conservation, the terms included in the underbraces cancel, leaving

$$\boxed{\rho \frac{De}{Dt} = - \frac{\partial Q_i}{\partial x_i} - p \frac{\partial q_i}{\partial x_i} + \frac{\partial (\tau_{ij} q_i)}{\partial x_j}} \quad (4.1.5)$$

We must now add Fick's law of heat conduction

$$Q_i = -K \frac{\partial T}{\partial x_i} \quad (4.1.6)$$

where K is the heat conductivity, and the following equations of state

$$e = e(p, T) \quad (4.1.7)$$

$$\rho = \rho(p, T) \quad (4.1.8)$$

Now there are 10 unknowns q_i, Q_i, ρ, p, e, T , and 10 equations: 1 from (4.1.1), 3 from (4.1.2), 1 from (4.1.5), 3 from (4.1.6), 1 from (4.1.7) and 1 from (4.1.8).

4.1.2 Equations of state:

For a perfect gas:

$$e = C_v T \quad (4.1.9)$$

$$p = \rho RT, \quad \text{where } R = C_p - C_v. \quad (4.1.10)$$

The specific heats C_v (constant volume) and C_p (constant pressure) are measured in Joules/kg-dyne.

For a liquid:

$$e = CT \quad (4.1.11)$$

$$\rho = \bar{\rho}_o \left(\bar{T}_o, \bar{p}_o \right) + \left. \frac{\partial \rho}{\partial p} \right|_T \Delta p + \left. \frac{\partial \rho}{\partial T} \right|_p \Delta T + \dots$$

where $\bar{\rho}_o, \bar{T}_o, \bar{p}_o$ are some constant reference density, temperature, and pressure respectively, while $\Delta p = p - \bar{p}_o$ and $\Delta T = T - \bar{T}_o$ are the variations in pressure and temperature. These variations are usually small in environmental problems. We define the thermal expansion coefficient β by :

$$\beta = - \left. \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right|_p = \left. \frac{1}{V} \frac{\partial V}{\partial T} \right|_p \quad (4.1.12)$$

where V denotes the specific volume (volume per unit mass), and the bulk modulus $1/\varepsilon$ by

$$\varepsilon = - \left. \frac{1}{\rho} \frac{\partial \rho}{\partial p} \right|_T = - \left. \frac{1}{V} \frac{\partial V}{\partial p} \right|_T \quad (4.1.13)$$

$$\rho \cong \bar{\rho}_o (1 - \beta \Delta T + \varepsilon \Delta p) \quad (4.1.14)$$

For liquids

$$O(\beta) \sim 10^{-3}/^\circ K, \quad O(\varepsilon) \sim 10^{-6}/atm$$

Rewrite Eqn. (4.1.5)

$$\rho \frac{De}{Dt} + \rho q_i \frac{Dq_i}{Dt} = \rho q_i f_i + q_i \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial}{\partial x_i} K \frac{\partial T}{\partial x_i} - p \frac{\partial q_i}{\partial x_i} + \tau_{ij} \frac{\partial q_i}{\partial x_j} \quad (4.1.15)$$

In summary, we have, for a perfect gas

$$\boxed{\rho C_v \frac{DT}{Dt} = -p \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i} K \frac{\partial T}{\partial x_i} + \tau_{ij} \frac{\partial q_i}{\partial x_j}} \quad (4.1.16)$$

and for an incompressible liquid

$$\boxed{\rho C \frac{DT}{Dt} = \frac{\partial}{\partial x_i} K \frac{\partial T}{\partial x_i} + \tau_{ij} \frac{\partial q_i}{\partial x_j}} \quad (4.1.17)$$

Recall that

$$\Phi = \tau_{ij} \frac{\partial q_i}{\partial x_j} \quad (4.1.18)$$

is the rate of viscous dissipation. Thus the rate of change in internal energy is equal to the sum of rate of pressure working to compress the fluid, viscous dissipation, and heat diffusion. More will be said about the incompressibility of liquid later.

4.2 Approximations for small temperature variation

4.2.1 Mass conservation and almost incompressibility :

Recall the law of mass conservation:

$$-\frac{1}{\rho} \frac{D\rho}{Dt} = \nabla \cdot \vec{q}$$

Let the time scale be L/U . The left-hand-side is of the order $\frac{U}{L} \frac{\Delta\rho}{\rho}$ while the right-hand-side is $\frac{U}{L}$. For $\Delta T = O(10^\circ C)$, their ratio is

$$\frac{\Delta\rho}{\rho} \sim \frac{\Delta T}{T} \sim \frac{10^\circ K}{300^\circ K} \ll 1$$

Therefore,

$$\nabla \cdot \vec{q} = 0. \quad (4.2.1)$$

The fluid is approximately incompressible even if $\Delta T \neq 0$.

4.2.2 Momentum conservation and Boussinesq approximation

In static equilibrium $\vec{q}_o \equiv 0$. Therefore,

$$-\nabla p_o + \vec{f} \rho_o = 0. \quad (4.2.2)$$

Let $p = p_d + p_o$ where p_d is the dynamic pressure

$$\rho = \rho_d + \rho_o$$

$$-\nabla p + \rho \vec{f} = -\nabla p_o + \rho_o \vec{f} - \nabla p_d + (\rho - \rho_o) \vec{f}$$

Therefore,

$$\rho \frac{D\vec{q}}{Dt} = -\nabla p_d + \nabla \cdot \bar{\bar{\tau}} + \underbrace{(\rho - \rho_o) \vec{f}}_{\text{buoyancy force}} \quad (4.2.3)$$

Now

$$\rho = \bar{\rho}_o [1 - \beta(\Delta T_o + \Delta T_d)] \quad (4.2.4)$$

Hence

$$\rho_o = \bar{\rho}_o(1 - \beta\Delta T_o), \quad \rho_d = -\bar{\rho}_o\beta\Delta T_d,$$

and

$$(\rho - \rho_o)\vec{f} = -\bar{\rho}_o(-g)\beta\Delta T_d\vec{k} = \bar{\rho}_og\beta\Delta T_d\vec{k} \quad (4.2.5)$$

For mildly varying ρ_o and small $\rho - \rho_o$, we ignore the variation of density and approximate ρ_o by a constant everywhere, except in the body force. This is called the **Boussinesq approximation**. Thus

$$\bar{\rho}_o\frac{D\vec{q}}{Dt} = -\nabla p_d + \nabla \cdot \bar{\tau} + \bar{\rho}_og\beta\Delta T_d\vec{k} \quad (4.2.6)$$

where

$$\bar{\rho}_o = \rho_o(z = 0)$$

4.2.3 Total energy

Using Eqn. (4.2.1) in Eqn. (4.1.17) and the Boussinesq approximation

$$\bar{\rho}_oC\frac{DT}{Dt} = \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} + \Phi \quad (4.2.7)$$

Here T is the total temperature (static + dynamic).

Now

$$\frac{\Phi}{\bar{\rho}_oC\frac{DT}{Dt}} \sim \frac{\mu U^2/L^2}{\bar{\rho}_oC\frac{U\Delta T}{L}} \sim \frac{\mu}{\bar{\rho}_oUL} \frac{U^2}{C\Delta T} = \frac{E}{Re}$$

where

$$E = \frac{U^2}{C\Delta T} = \text{Eckart No.}, \quad Re = \frac{\rho UL}{\mu} = \text{Reynolds No.}$$

In environmental problems, $\Delta T \sim 10^\circ K$, $L \sim 10\text{ m}$, $U \sim 1\text{ m/sec}$, the last two columns of

Table 4.1: Typical values E/Re for air and water

	Water	Air
C (erg/s-gr- $^\circ K$)	4×10^7	10^7
K (ergs-cm- $^\circ K$)	0.6×10^5	0.3×10^5
ν (cm^2/s)	10^2	2×10^{-2}
β ($1/^\circ K$)	10^{-3}	1/300
E	0.25×10^{-2}	10^{-4}
Re	10^5	0.5×10^5

Table 4.1 is obtained. Hence Φ is negligible, and

$$\bar{\rho}_oC\frac{DT}{Dt} = \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i} \quad (4.2.8)$$

Only convection and diffusion are dominant. This is typical in natural convection problems.

Remark 1. In many engineering problems (aerodynamics, rocket reentry, etc.), heat is caused by frictional dissipation in the flow, therefore, Φ is important. These are called **forced convection** problems. In environmental problems, flow is often the result of heat addition. Here we have the **natural convection**.

Remark 2: Since \bar{T} appears as a derivative only, only the variation of T , i.e., the difference $T - \bar{T}_o$ matters, where \bar{T}_o is a reference temperature.

Remark 3: In turbulent natural convection

$$u = \bar{u} + u' \quad T = \bar{T} + T' \quad (4.2.9)$$

Averaging Eqn. (4.2.8)

$$\bar{\rho}_o c \frac{D\bar{T}}{Dt} = - \underbrace{\bar{\rho}_o c \frac{\partial \overline{u'_i T'}}{\partial x_i}}_{\text{heat flux by turbulence}} + \frac{\partial}{\partial x_i} K \frac{\partial \bar{T}}{\partial x_i} \quad (4.2.10)$$

If the correlation term is modeled as eddy diffusion, the form would be similar to (4.2.8).