Notes on
1.63 Advanced Environmental Fluid Mechanics
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7.3 The Shallow-Water Approximation

The continuity equation reads

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{U}{L}}+\underbrace{\frac{\partial v}{\partial y}}_{\frac{U}{L}}+\underbrace{\frac{\partial w}{\partial z}}_{\frac{W}{D}}=0$$

If the horizontal length scale of motion L is much greater than the sea depth D, i.e., shallow water,

$$\frac{D}{L}\ll 1 \tag{7.3.1}$$

we have

$$W=\frac{DU}{L}\ll U \tag{7.3.2}$$

Momentum conservation requires

$$\underbrace{\frac{\partial u}{\partial t}}_{\frac{U}{T}}+(\underbrace{u\frac{\partial u}{\partial x}}_{\frac{U^2}{L}}+\underbrace{v\frac{\partial u}{\partial y}}_{\frac{U^2}{L}}+\underbrace{w\frac{\partial u}{\partial z}}_{\frac{U^2}{L}})-\underbrace{fv}_{fU}=\underbrace{\frac{1}{\rho}\frac{\partial p_d}{\partial x}}_{\frac{P}{\rho L}}$$

$$\frac{\partial v}{\partial t}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+fu=-\frac{1}{\rho}\frac{\partial p_d}{\partial y},$$

$$\underbrace{\frac{\partial w}{\partial t}}_{\frac{W}{T}}+(\underbrace{u\frac{\partial w}{\partial x}}_{\frac{UW}{L}}+\underbrace{v\frac{\partial w}{\partial y}}_{\frac{UW}{L}}+\underbrace{w\frac{\partial w}{\partial z}}_{\frac{UW}{L}})=\underbrace{\frac{1}{\rho}\frac{\partial p_d}{\partial z}}_{\frac{P}{\rho D}}$$

where p_d stands for the dynamic pressure

$$p=-\rho gz+p_d$$

Examine first the horizontal momentum balance. In order that pressure gradient forces the rotating flow, the pressure gradient has to balance either the Coriolis force (rotation important), assuming that the Rossby number is no greater than $O(1)$. The scale of pressure is

$$P=\frac{\rho UL}{T},\quad \text{or}\quad \rho UfL$$

From the vertical momentum we find, if $P = \rho UL/T$,

$$\frac{\frac{\partial w}{\partial t}}{\frac{1}{\rho}\frac{\partial_d}{\partial z}} \sim \frac{\frac{DU}{LT}}{\frac{\rho UL/T}{\rho D}} \sim \frac{D^2}{L^2}$$

and if $P = \rho UfL$,

$$\frac{\frac{\partial w}{\partial t}}{\frac{1}{\rho}\frac{\partial_d}{\partial z}} \sim \frac{\frac{DU}{LT}}{\frac{UfL}{\rho D}} \sim \frac{D^2}{L^2}\frac{1}{Tf}$$

Since the time scale of interest is of a day or so, $fT = O(1)$. We conclude that the vertical pressure gradient is practially zero with error of order $D^2/L^2 \ll 1$, implying that

$$p_d \simeq \rho g \eta \tag{7.3.3}$$

or the total pressure is hydrostatic

$$p_{\text{total}} \simeq \rho g(\eta - z) \tag{7.3.4}$$

Thus,

$$\frac{\partial p_d}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \qquad \frac{\partial p_d}{\partial y} = \rho g \frac{\partial \eta}{\partial y} \tag{7.3.5}$$

We now show by a formal perturbation scheme for horizontal bottom, that u, v are independent of z to the leading order or approximation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.3.6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.3.7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.3.8}$$

On the bottom

$$w = 0 \tag{7.3.9}$$

On the free surface

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w, \quad z = \eta \tag{7.3.10}$$

Assuming the following expansions:

$$u = u_0 + \frac{z+h}{L}u_1 + \frac{(z+h)^2}{2L^2}u_2 + \cdots \tag{7.3.11}$$

$$v = v_0 + \frac{z+h}{L}v_1 + \frac{(z+h)^2}{2L^2}v_2 + \cdots \tag{7.3.12}$$

$$w = \frac{z+h}{L} w_1 + \frac{(z+h)^2}{2L^2} w_2 + \cdots \tag{7.3.13}$$

where u_n, v_n, w_n are independent of z. From the x momentum equation we get from (7.3.7)

$$\begin{aligned}
&\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - f v_0 + g \frac{\partial \eta}{\partial x} \\
&\quad + \frac{(z+h)}{L} \left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_1}{\partial y} + \frac{w_1 u_1}{L} - f v_1 \right) \\
&\quad + \left(\frac{z+h}{L} \right)^2 (\cdots) + \cdots = 0
\end{aligned} \tag{7.3.14}$$

with a similar equation for the y momentum. Separating the zeroth power of $(z+h)/L$ we get

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - f v_0 = -g \frac{\partial \eta}{\partial x} \tag{7.3.15}$$

$$\frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} + v_0) \frac{\partial v_0}{\partial y} + f u_0 = -g \frac{\partial \eta}{\partial y} \tag{7.3.16}$$

Thus u, v are depth-independent to the leading order. Also from continuity,

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{w_1}{L} = 0 \tag{7.3.17}$$

and from the free surface condition

$$\frac{\partial \eta}{\partial t} + u_0 \frac{\partial \eta}{\partial x} = \frac{\eta + h}{L} w_1 \tag{7.3.18}$$

Hence after eliminating w_1,

$$\frac{\partial \eta}{\partial t} + \frac{\partial[(\eta+h)u_0]}{\partial x} + \frac{\partial[(\eta+h)v_0]}{\partial y} = 0 \tag{7.3.19}$$

In summary, for shallow seas,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -\frac{1}{\rho} \frac{\partial \eta}{\partial x} \tag{7.3.20}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -\frac{1}{\rho} \frac{\partial \eta}{\partial y} \tag{7.3.21}$$

to leading order.

If Rossby number is small, then the convective inertia is negligible; the momentum equations reduce to :

$$\frac{\partial u}{\partial t} - f v = -g \frac{\partial \eta}{\partial x} \tag{7.3.22}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.3.23}$$

where are linear.

Take for estimates, $U = 0.1, 1m/s, \Omega = 2.31 \times 10^{-5}, L = 100, 1000km = 10^5, 10^6 m$, Rossby number $U/2\Omega L = 0.043$. We leave it as an exercise to work out the equations for $h(x, y)$ with a small slope.

7.3.1 Geostrophic motion

For steady flow at small Rossby number,

$$H = \eta + h \simeq h$$

the momentum equations reduce to

$$\begin{aligned} -fv &= -g\frac{\partial \eta}{\partial x} \\ fu &= -g\frac{\partial \eta}{\partial y}. \end{aligned}$$

Thus, Coriolis force and pressure gradient are in balance

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} \qquad v = \frac{g}{f}\frac{\partial \eta}{\partial x}$$

implying

$$u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0 \tag{7.3.24}$$

or

$$\vec{q} \cdot \nabla \eta = 0.$$

Physically along a streamlines, the free surface height remains constant. Hence, the surface contours are parallel to the streamlines and to isobars. This state is called **geostrophic**.

7.3.2 Remark: Depth-integrated mass conservation

The depth integrated mass conservation (7.3.25) is a general an exact result which holds for variable depth as well. On the seabed $z = -h(x, y)$, vanishing of the normal veocity requires

$$w = -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y}.$$

On the free surface $z = \eta(x, y, t)$, the kinematic boundary condition reads,

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \qquad z = \eta$$

Integrating the continuity equation and using Leibniz's rule

$$
\begin{aligned}
0 &= \int_{-h}^{\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz \\
&= [w]_{-h}^{\eta} + \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} v dz \\
&\quad + \frac{\partial \eta}{\partial x} u(\eta) + \frac{\partial \eta}{\partial y} v(\eta) - \frac{\partial h}{\partial x} u(-h) - \frac{\partial h}{\partial y} v(-h).
\end{aligned}
$$

Define the total depth to be $H = \eta + h$

$$
0 = \left(w - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y} \right)_{\eta} - \left(w + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial x} \right)_{-h} + \frac{\partial H u}{\partial x} + \frac{\partial (H v)}{\partial y}
$$

Using the boundary conditions we get

$$
\frac{\partial H}{\partial t} + \frac{\partial H u}{\partial x} + \frac{\partial H v}{\partial y} = 0. \tag{7.3.25}
$$