Notes on
1.63 Advanced Environmental Fluid Mechanics
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chapter

7.2 Vorticity in inviscid rotating fluids – Taylor -Proudman theorem

Ignoring viscosity, let $\vec{\zeta} = \nabla \times \vec{q}$ and use the identity

$$\vec{\zeta} \times \vec{q} = \vec{q} \cdot \nabla \vec{q} - \nabla \frac{|\vec{q}|^2}{2}$$

The mommentum equation can be written :

$$\frac{\partial \vec{q}}{\partial t} + \vec{\zeta} \times \vec{q} + 2\vec{\Omega} \times \vec{q} = -\frac{\nabla p}{\rho} + \nabla \left(\phi - \frac{|\vec{q}|^2}{2} \right) \tag{7.2.1}$$

Taking the curl of the above equation:

$$\frac{\partial \vec{\zeta}}{\partial t} + \nabla \times \left((2\vec{\Omega} + \vec{\zeta}) \times \vec{q} \right) = \frac{\nabla \rho \times \nabla p}{\rho^2}$$

Using the identity

$$\nabla \times \left(\vec{A} \times \vec{B} \right) = \vec{A} \nabla \cdot \vec{B} - \vec{B} \nabla \cdot \vec{A} + \vec{B} \cdot \nabla \vec{A} - \vec{A} \cdot \nabla \vec{B}$$

we get

$$\nabla \times \left\{ (2\vec{\Omega} + \vec{\zeta}) \times \vec{q} \right\} = -\vec{q} \nabla \cdot (2\vec{\Omega} + \vec{\zeta}) + (2\vec{\Omega} + \vec{\zeta}) \nabla \cdot \vec{q} + \vec{q} \cdot \nabla (2\vec{\Omega} + \vec{\zeta}) - (2\vec{\Omega} + \vec{\zeta}) \cdot \nabla \vec{q}$$

The first term on the right vanishes because $\vec{\Omega} =$ constant and the divergence of curl is zero; the second vanishes for imcompressible fluids. Let $\vec{\zeta_a} = \vec{\zeta} + 2\vec{\Omega} =$ absolute vorticity

$$\frac{D\vec{\zeta}}{Dt} = \frac{\partial \vec{\zeta}}{\partial t} + \vec{q} \cdot \nabla \vec{\zeta} = \vec{\zeta_a} \cdot \nabla \vec{q} + \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{7.2.2}$$

If the Rossby number is small (slow flow)

$$\epsilon = \text{Rossby No.} = \frac{u}{2\Omega L} \ll 1$$

and if $\rho =$ constant, then

$$\frac{\zeta}{2\Omega} \sim \frac{u}{2\Omega L} \ll 1$$

and

$$2\vec{\Omega} \cdot \nabla \vec{q} = 0 \tag{7.2.3}$$

Th eflow is two dimensional and does not vary along the rotation vector. Let Ω be parallel to the z axis, then

$$\frac{\partial \vec{q}}{\partial z} = 0 \tag{7.2.4}$$

This has been proven edperimentally by Taylor, see sketch in Figure 7.2.1.

Theorem 1 *Taylor-Proudman theorem : A steady and slow flow in a rotating fluid is two-dimensional in the plane perpendicular to the vector of angular velocity.*

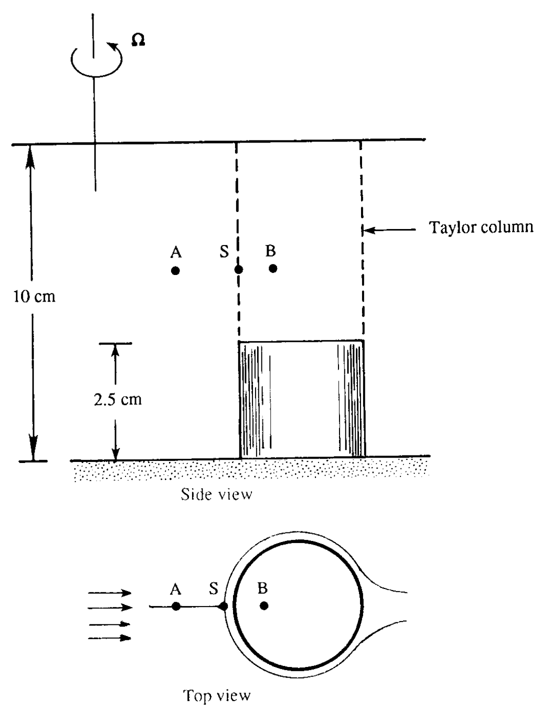

Fig. 13.6 G. I. Taylor's experiment in a strongly rotating flow of a homogeneous fluid.

Figure 7.2.1: Taylor's experiment showing the Taylor column above a truncated cylinder in a rotating fluid. The large container with water rotates but the cylinder is fixed in space. From Kundu.