

Notes on  
**1.63 Advanced Environmental Fluid Mechanics**  
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 1-7vorti.tex

## 1.7 Vorticity Theorem for a viscous fluid

Consider an incompressible fluid.

$$\nabla \cdot \vec{q} = 0 \quad (1.7.1)$$

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \nabla \vec{q} = \vec{f} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q} \quad (1.7.2)$$

where

$$\mu = \rho\nu, \quad \nu = \text{kinematic viscosity.} \quad (1.7.3)$$

Note the following identity in index form:

$$\begin{aligned} q_j \frac{\partial q_i}{\partial x_j} &= q_j \left( \frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i} \right) + q_j \frac{\partial q_j}{\partial x_i} \\ &= 2q_j \Omega_{ij} + \frac{\partial}{\partial x_i} \left( \frac{1}{2} q_j q_j \right) \\ &= \left( \vec{\zeta} \times \vec{q} \right)_i + \left( \nabla \frac{\vec{q}^2}{2} \right)_i. \end{aligned}$$

where  $\vec{\zeta} = \nabla \times \vec{q}$ . Hence, in vector form, the convective inertia term can be written as

$$\vec{q} \cdot \nabla \vec{q} = \nabla \frac{\vec{q}^2}{2} + \vec{\zeta} \times \vec{q} \quad (1.7.4)$$

We can now rewrite the momentum equation (1.7.2) by using (1.7.4)

$$\frac{\partial \vec{q}}{\partial t} + \nabla \frac{\vec{q}^2}{2} + \vec{\zeta} \times \vec{q} = \vec{f} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q}. \quad (1.7.5)$$

The equation governing the transport of vorticity follows by taking the curl of (1.7.5). For a conservative body force such as gravity,

$$\nabla \times \vec{f} \equiv 0 \quad (1.7.6)$$

Recall the vector identities:

$$\nabla \times \nabla S = 0 \quad (1.7.7)$$

for any scalar  $S$ ,

$$\nabla \cdot \nabla \times \vec{T} = 0 \quad (1.7.8)$$

for any vector  $\vec{T}$ , and

$$\nabla \times (\vec{\zeta} \times \vec{q}) = \vec{q} \cdot \nabla \zeta - \vec{\zeta} \cdot \nabla \vec{q} + \zeta \nabla \cdot \vec{q} - \vec{q} \nabla \cdot \vec{\zeta} \quad (1.7.9)$$

for any cross product of two vectors. In (1.7.9), let  $\vec{q}$  be the fluid velocity and  $\vec{\zeta}$  be the vorticity. Then the third term on the right vanishes on account of incompressibility, and the last vanishes because  $\vec{\zeta} = \nabla \times \vec{q}$ .

Finally, we get the vorticity transport equation:

$$\frac{\partial \vec{\zeta}}{\partial t} + \vec{q} \cdot \nabla \vec{\zeta} = \vec{\zeta} \cdot \nabla \vec{q} + \nu \nabla^2 \vec{\zeta} + \frac{\nabla \rho \times \nabla p}{\rho^2} \quad (1.7.10)$$

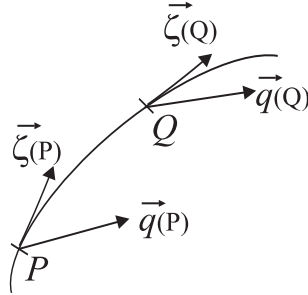


Figure 1.7.1: Two neighboring points on a vortex line

We now examine the physical meaning of each term in (1.7.10):

- $\frac{D\vec{\zeta}}{Dt}$  is the rate of change of vorticity following the moving fluid.
- $\nu \nabla^2 \vec{\zeta}$  represents the diffusion of vorticity by viscosity.
- $\vec{\zeta} \cdot \nabla \vec{q}$ : Let us rewrite Eq. (1.7.10)

$$\frac{1}{|\zeta|} \frac{D\vec{\zeta}}{Dt} = \frac{\vec{\zeta}}{|\zeta|} \cdot \nabla \vec{q} + \dots$$

Define a vortex line as the curve which is tangent to all the local vorticity vectors. Then

$$\frac{\vec{\zeta}}{|\zeta|} = \vec{e}_\zeta$$

is the local unit tangent to the vortex line. Let  $s$  be the arc-length along the vortex line:

$$\frac{\vec{\zeta}}{|\zeta|} \cdot \nabla \vec{q} = \vec{e}_\zeta \cdot \nabla \vec{q} = \frac{d\vec{q}}{ds} = \frac{d\vec{q}_\parallel}{ds} + \frac{d\vec{q}_\perp}{ds},$$

where  $\vec{q}_\parallel$  denotes the fluid velocity along the vortex line while  $\vec{q}_\perp$  denotes the velocity normal to the vortex line. Hence  $\frac{d\vec{q}_\parallel}{ds}$  is the rate of stretching of the vortex line, while  $\frac{d\vec{q}_\perp}{ds}$  is the rate of rotation of the vortex line. It follows that  $\vec{\zeta} \cdot \nabla \vec{q}$  represents the rate of production of vorticity by stretching and turning of the vortex line.

- Finally  $\nabla \rho \times \nabla p / \rho^2$  is the vorticity source due to density nonuniformity; this is important in stratified fluids.

In summary, following the fluid motion, the increase in vorticity is caused by production due to stretching and turning of the vortex lines, diffusion by viscosity and density inhomogeneity.