Notes on
1.63 Advanced Environmental Fluid Mechanics
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March 10, 2001
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1.6 Relations between stress and rate-of-strain tensors

When the fluid is at rest on a macroscopic scale, no tangential stress acts on a surface. There is only the normal stress, i.e., the pressure $-p\delta_{ij}$ which is thermodynamic in origin, and is maintained by molecular collisions. Denoting the additional stress by τ_{ij} which is due to the relative motion on the continuum scale,

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{1.6.1}$$

The second part is called the viscous stress τ_{ij} and must depend on gradients of velocity,

$$\frac{\partial q_i}{\partial x_j}, \frac{\partial^2 q_i}{\partial x_j \partial x_k} \cdots.$$

1.6.1 Newtonian fluid

For many fluids in nature such as air and water, the relation between τ_{ij} and $\partial q_i/\partial x_i$ are linear under most circumstances. Such fluids are called Newtonian. For second-rank tensors, the most general linear relation is,

$$\tau_{ij} = C_{ij\ell m}\frac{\partial q_\ell}{\partial x_m}. \tag{1.6.2}$$

where $C_{ij\ell m}$ is a coefficient tensor of rank 4. In principle there are $3^4 = 81$ coefficients. It can be shown (Spain, *Cartesian Tensors*) that in an isotropic fluid the fourth-rank tensor is of the following form:

$$C_{ij\ell m} = \lambda\delta_{ij}\delta_{\ell m} + \mu(\delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}) \tag{1.6.3}$$

Eighty one coefficients in $C_{ij\ell m}$ reduce to two: λ and μ, and

$$\tau_{ij} = \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) + \lambda\frac{\partial q_\ell}{\partial x_\ell}\delta_{ij}. \tag{1.6.4}$$

where μ, λ are viscosity coefficients depending empirically on temperature.

Note that the velocity gradient is made up of two parts

$$\frac{\partial q_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right)$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \tag{1.6.5}$$

is the rate of strain tensor, and

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right) \tag{1.6.6}$$

is the vorticity tensor. Note also that (1.6.4) depends only on the rate of strain but not on vorticity. This is reasonable since a fluid in rigid-body rotation should not experience any viscous stress. In a rigid-body rotation with angular velocity ω, the fluid velocity is

$$\vec{q} = \vec{\omega} \times \vec{r} \qquad q_i = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

The vorticity components are not zero; for example,

$$2\Omega_{12} = \frac{\partial q_1}{\partial x_2} - \frac{\partial q_2}{\partial x_1} \quad = \quad \frac{\partial}{\partial x_2}(\omega_2 x_3 - \omega_3 x_2) - \frac{\partial}{\partial x_1}(\omega_3 x_1 - \omega_1 x_3) = -2\omega_3.$$

Hence τ_{ij} cannot depends on Ω_{ij} and only on e_{ij}.

The trace of σ_{ij} is

$$\sigma_{ii} = (2\mu + 3\lambda)\frac{\partial q_i}{\partial x_i} = (3\lambda + 2\mu)\nabla \cdot \vec{q}.$$

where $k = 3\lambda + 2\mu =$ is called the bulk viscosity.

For incompressible fluids $\nabla \cdot \vec{q} = 0$; the viscous stress tensor is

$$\sigma_{ij} = \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}\right) \tag{1.6.7}$$

The total stress tensor is therefore

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \tag{1.6.8}$$

The governing equations for an incompressible Newtonian fluid may now be summarized:

$$\frac{D\rho}{Dt} = 0, \quad \text{(incompressibility)} \tag{1.6.9}$$

$$\frac{\partial q_i}{\partial x_i} = 0, \quad \text{(continuity)} \tag{1.6.10}$$

$$\rho\frac{Dq_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 q_i}{\partial x_j \partial x_j} + \rho f_i, \quad \text{(momentum conservation)} \tag{1.6.11}$$

after using continuity. The last equation (1.6.11) and sometimes the set of equations (1.6.11), (1.6.9) and (1.6.10), is called the Navier-Stokes equation(s). Now we have just five scalar equations for five unknowns ρ, p, and q_i. Boundary and initial conditions must be further specified. For example on the surface of a stationary rigid body, no slippage is allowed, so that

$$q_i = 0, \quad \text{on a rigid stationary surface} \tag{1.6.12}$$

1.6.2 Non-Newtonian fluids

Many fluids such as toothpaste, gel, honey, heavy oil (DNAPL), etc., flows like a fluid if the shear stress is above a critical value, and behaves like a solid if below. Of geological interest is the mud which is a mixture of water with highly cohesive clay particles. From volcanic eruption, lava can mix with rain, melting snow, or lake water to form mud, which flows down the hill slope, carries along stones, trees and other debris, to cause severe damages. In some mountainous areas, heavy rainfall infiltrates the top soil and causes mud to slide and flow into rivers to form hyper concentrated fluid-mud. These fluids are called non-Newtonian since the relation between σ_{ij} and e_{ij} is nonlinear.

We shall only discuss a special model of non-Newtonian fluid, i.e., the Bingham plastic model. For simple shearing flow $u = u(y)$, the constitutive relation for a Bingham plastic is

$$\begin{aligned} \frac{\partial u}{\partial y} &= 0, \quad \tau \leq \tau_c; \\ \frac{\partial u}{\partial y} &= \frac{1}{\mu}(\tau - \tau_c), \quad \tau > \tau_c \end{aligned} \tag{1.6.13}$$

where τ_c is called the yield stress and μ the Bingham viscosity, both of which depend on the clay concentration C.

In three dimensions, the Bingham model can be generalized by introducing the second invariants of the stress and rate-of-strain tensors. The second invariant of the viscous stress tensor is

$$\begin{aligned} II_T &\equiv \frac{1}{2}\left[\tau_{ij}\tau_{ij} - (\tau_{kk})^2\right] \\ &= \tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2 - (\tau_{11}\tau_{22} + \tau_{11}\tau_{33} + \tau_{22}\tau_{33}) \end{aligned} \tag{1.6.14}$$

Similarly the second invariant of the rate of strain tensor is

$$\begin{aligned} II_E &\equiv \frac{1}{2}\left[e_{ij}e_{ij} - (e_{kk})^2\right] \\ &= e_{12}^2 + e_{23}^2 + e_{31}^2 - (e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33}) \\ &= \frac{1}{4}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] \\ &\quad - \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial w}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right) \end{aligned} \tag{1.6.15}$$

The Bingham plastic law is then

$$\begin{aligned} e_{ij} &= 0, \quad \text{if} \quad \sqrt{II_T} < \tau_c, \\ \tau_{ij} &= 2\mu e_{ij} + \tau_c \frac{e_{ij}}{\sqrt{II_E}}, \quad \text{if} \quad \sqrt{II_T} \geq \tau_c. \end{aligned} \tag{1.6.16}$$

This is due to Hohenemser and Prager (1936).

In simple shear $u = u(y)$, the only non-zero components of τ_{ij} and e_{ij} are τ_{xy} and e_{xy}. The Bingham law reduces to

$$\begin{aligned} e_{xy} &= 0, \quad \text{if} \quad |\tau_{xy}| < \tau_c, \\ \tau_{xy} &= 2\mu e_{xy} + \tau_c, \quad \text{if} \quad |\tau_{xy}| \geq \tau_c. \end{aligned} \tag{1.6.17}$$

In other words,

$$\begin{aligned} \frac{\partial u}{\partial y} &= 0, \quad \text{if} \quad |\tau_{xy}| < \tau_c, \\ \tau_{xy} &= \mu \frac{\partial u}{\partial y} + \tau_c \text{sgn}\left(\frac{\partial u}{\partial y}\right), \quad \text{if} \quad |\tau_{xy}| \geq \tau_c. \end{aligned} \tag{1.6.18}$$

We shall examine some examples later.