I-campus project

School-wide Program on Fluid Mechanics

Modules on Waves in fluids

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CHAPTER SIX

INTERNAL WAVES IN A STRATIFIED FLUID

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1 Two-dimensional Internal waves in inviscid stratified fluid

Due to seaonal changes of temperature, the density of water or atmosphere can have significant variations in the vertical direction. Variation of salt content can also lead to density stratification. Freshwater from rivers can rest on top of the sea water. Due to the small diffusivity, the density contrast remains for a long time.

Consider a calm and stratified fluid with a static density distribution $\overline{\rho}_o(z)$ which decreases with height (z). If a fluid parcel is moved from the level z upward to $z+\zeta$, it is surrounded by lighter fluid of density $\overline{\rho}(z+dz)$. The upward buoyancy force per unit volume is

$$g(\overline{\rho}(z+\zeta)-\overline{\rho}(z)) \approx g\frac{d\overline{\rho}}{dz}\zeta$$

and is negative. Applying Newton's law to the fluid parcel of unit volume

$$\overline{\rho}\frac{d^2\zeta}{dt^2} = g\frac{d\overline{\rho}}{dz}\zeta$$

or

$$\frac{d^2\zeta}{dt^2} + N^2\zeta = 0 \tag{1.1}$$

where

$$N = \left(-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}\right)^{1/2} \tag{1.2}$$

is called the Brunt-Väisälä frequency. Ths elementary consideration shows that once a fluid is displaced from its equilibrium position, gravity and density gradient provides restoring force to enable oscillations. In general ther must be horizontal nonunifomities, hence waves are possible.

We start from the exact equations for an inviscid and incompressible fluid with variable density.

For an imincompuresibel fluid the density remains constant as the fluid moves,

$$\rho_t + \mathbf{q} \cdot \nabla \rho = 0 \tag{1.3}$$

where $\mathbf{q} = (u, w)$ is the velocity vector in the vertical plane of (x, z). Conservation of mass requires that

$$\nabla \cdot \mathbf{q} = 0 \tag{1.4}$$

The law of momentum conservation reads

$$\rho(\mathbf{q}_t + \mathbf{q} \cdot \nabla \mathbf{q}) = \nabla p - \rho g \mathbf{e}_z \tag{1.5}$$

and $\mathbf{e}_z$ is the unit vector in the upward vertical direction.

1.1 Linearized equations

Consider small disturbances

$$p = \overline{p} + p', \quad \rho = \overline{\rho}(z) + \rho', \quad \vec{q} = (u', w') \tag{1.6}$$

with

$$\overline{\rho} \gg \rho', \quad \overline{p} \gg p' \tag{1.7}$$

and u', v', w' are small. Linearizing by omitting quadratically small terms associated with the fluid motion, we get

$$\rho'_t + w'\frac{d\overline{\rho}}{dz} = 0. \tag{1.8}$$

$$u'_x + w'_z = 0 \tag{1.9}$$

$$\overline{\rho} u'_t = -p'_x \tag{1.10}$$

$$\overline{\rho} w'_t = -\overline{p}_z - p'_z - g\overline{\rho} - g\rho' \tag{1.11}$$

In the last equation the static part must be in balance

$$0 = -\overline{p}_z - g\overline{\rho}, \tag{1.12}$$

hence

$$\overline{p}(z) = \int_0^z \bar{\rho}(z)dz. \tag{1.13}$$

The remaining dynamically part must satisfy

$$\overline{\rho} w'_t = -p'_z - g\rho' \tag{1.14}$$

Upon elminating p' from the two momentum equations we get

$$\frac{d\overline{\rho}}{dz} u'_t + \overline{\rho}(u'_z - w'_x)_t = g\rho'_x \tag{1.15}$$

Eliminating ρ' from (1.8) and (1.15) we get

$$\frac{d\overline{\rho}}{dz} u'_{tt} + \overline{\rho}(u'_z - w'_x)_{tt} = g\rho'_{xt} = -g\frac{d\overline{\rho}}{dz} w'_x \tag{1.16}$$

Let us introduce the disturbance stream function ψ:

$$u' = \psi_z, \quad w' = -\psi_x \tag{1.17}$$

It follows from (1.16) that

$$\overline{\rho}\left(\psi_{xx} + \psi_{zz}\right)_{tt} = \frac{d\overline{\rho}}{dz}\left(g\psi_{xx} - \psi_{ztt}\right) \tag{1.18}$$

by virture of Eqns. (1.8) and (1.17). Note that

$$N = \sqrt{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}} \tag{1.19}$$

is the Brunt-Väisälä frequency. In the ocean, density gradient is usally very small ($N \sim 5 \times 10^{-3}$ rad/sec). Hence $\overline{\rho}$ can be approximated by a constant reference value, say, $\rho_0 = \overline{\rho}(0)$ in (1.10) and (1.14) without much error in the inertia terms. However density variation must be kept in the buoyancy term associated with gravity, which is the only restoring force responsible for wave motion. This is called the **Boussinesq**

approximation and amounts to taking $\overline{\rho}$ to be contant in (Eq:17.1) only. With it (1.18) reduces to

$$\boxed{(\psi_{xx} + \psi_{zz})_{tt} + N^2(z)\,\psi_{xx} = 0.} \tag{1.20}$$

Note that because of linearity, u' and w' satisfy Eqn. (1.20) also, i.e.,

$$(w'_{xx} + w'_{zz})_{tt} + N^2\, w'_{xx} = 0 \tag{1.21}$$

etc.

1.2 Linearized Boundary conditions on the sea surface

Dynamic boundary condition : Total pressure is equal to the atmospheric pressure

$$(\overline{p} + p')_{z=\zeta} = 0. \tag{1.22}$$

On the free surface $z = \zeta$, we have

$$\overline{p} \approx -g \int_0^\zeta \overline{\rho}(0) dz = -g\overline{\rho}(0)\zeta$$

Therefore,

$$-\overline{\rho} g \zeta + p' = 0, \quad z = 0, \tag{1.23}$$

implying

$$-\overline{\rho} g \zeta_{xxt} + p'_{xxt} = 0, \quad z = 0. \tag{1.24}$$

Kinematic condition :

$$\zeta_t = w, \quad z = 0. \tag{1.25}$$

The left-hand-side of (1.24) can be written as

$$-\overline{\rho} g \zeta_{xxt} = -\overline{\rho} g\, w'_{xx}$$

Using 1.10, the right-hand-side of 1.24 may be written,

$$-p_{xxt} = \overline{\rho}\, u'_{xtt} = -\overline{\rho} w'_{ztt}$$

hence

$$w'_{ztt} - g w'_{xx} = 0, \quad \text{on} \quad z = 0. \tag{1.26}$$

Since $w' = -\psi_x$, ψ also satisfies the same boundary conditon

$$\boxed{\psi_{ztt} - g\psi_{xx} = 0, \quad \text{on} \quad z = 0.} \tag{1.27}$$

On the seabed, $z = -h(x)$ the normal velocity vanishes. For a horizontal bottom we have

$$\psi(x, -h, t) = 0. \tag{1.28}$$

1.3 Simple harmonic waves for finite N

Consider a horizontally propagating wave beneath the sea surface. Let

$$\psi = F(z)\, e^{\pm ikx} e^{-i\omega t}. \tag{1.29}$$

From Eqn. (1.21),

$$-\omega^2 \left(\frac{d^2F}{dz^2} - k^2 F\right) + N^2 \left(-k^2\right) F = 0$$

or,

$$\frac{d^2F}{dz^2} + \frac{N^2 - \omega^2}{\omega^2}\, k^2 F = 0 \quad z < 0. \tag{1.30}$$

On the (horizontal) sea bottom

$$F = 0 \qquad z = -h. \tag{1.31}$$

From Eqn. (1.27),

$$\frac{dF}{dz} - g\,\frac{k^2}{\omega^2}\, F = 0 \qquad z = 0. \tag{1.32}$$

Equations (1.30), (1.31) and (1.32) consititute an eigenvalue condition.

If $\omega^2 < N^2$, then F is oscillatory in z within the thermocline. Away from the thermocline, $\omega^2 > N^2$, W must decay exponentially. Therefore, the thermocline is a waveguide within which waves are trapped. Waves that have the greatest amplitude beneath the free surface is called internal waves.

Since for internal waves, $\omega < N$ while N is very small in oceans, oceanic internal waves have very low natural frequencies. For most wavelengths of practical interests $\omega^2 \ll gk$ so that

$$F \cong 0 \;\; \text{on } z = 0. \tag{1.33}$$

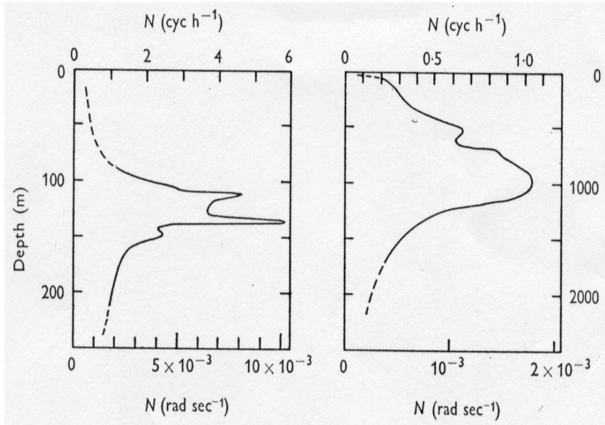

Figure 1: From Phillips, 1977

This is called the **rigid lid approximation**, which will be adopted in the following for simplicity.

With the rigid-lid approximation, the solution for F is

$$F = A \sin\left(k(z+h)\frac{\sqrt{N^2-\omega^2}}{\omega}\right) \tag{1.34}$$

where

$$kh\frac{\sqrt{N^2-\omega^2}}{\omega} = n\pi, \quad n = 1, 2, 3... \tag{1.35}$$

This is an eigen-value condition. For a fixed wave number k, it gives the eigen-frequencies,

$$\omega_n = \frac{N}{\sqrt{1+\left(\frac{n\pi}{kh}\right)^2}} \tag{1.36}$$

For a given wavenumber k, this dispersion relation gives the eigen-frequency ω_n. For a given frequency ω, it gives the eigen-wavenumbers k_n,

$$k_n = \frac{n\pi}{h}\frac{\omega}{\sqrt{N^2-\omega^2}} \tag{1.37}$$

For a simple lake with vertical banks and length L, $0 < x < L$, we must impose the conditions :

$$u' = 0, \quad \text{hence} \quad \psi = 0, \quad x = 0, L \tag{1.38}$$

The solution is

$$\psi = A \sin k_m x \, \exp(-i\omega_{nm}t) \sin\left[k_m(z+h)\frac{\sqrt{N^2-\omega_{nm}^2}}{\omega_{nm}}\right]. \tag{1.39}$$

with

$$k_m L = m\pi, \quad m = 1, 2, 3, ... \tag{1.40}$$

The eigen-frequencies are:

$$\omega_{nm} = \frac{N}{\sqrt{1 + \left(\frac{nL}{mh}\right)^2}} \tag{1.41}$$

1.4 Internal waves in a vertically unbounded fluid

Consider $N =$ constant, and denote by (α, β) the (x, z) components of the wave numberector $\vec{k}$ Let the solution be a plane wave in the vertical plane

$$\psi = \psi_0 \, e^{i(\alpha x + \beta z - \omega t)}$$

Then

$$\omega^2 = N^2 \frac{\alpha^2}{\alpha^2 + \beta^2}$$

or

$$\omega = \pm N \frac{\alpha}{k} \tag{1.42}$$

$$k^2 = \alpha^2 + \beta^2 \tag{1.43}$$

For a given frequency, there are two possible signs for α. Since the above relation is also even in β, there are four possible inclinations for the wave crests and troughs with respect to the vertical; the angle of inclination is

$$|\theta| = \cos^{-1} \frac{\omega}{N} \tag{1.44}$$

For $\omega < N$, $|\theta| < \pi/2$. There is no vertically propagating internal wave. This unique property of anisotropy has been verified in dramatic experiments by Mowbray and Stevenson. By oscillating a long cylinder at various frrequencies vertically in a stratified fluid, equal phase lines are only found along four beams forming "St Andrew's Cross", see figure (**??**) for $\omega/N = 0.7, \, 0.9$. It can be verified that angles are $|\theta| = 45°$ for $\omega/N = 0.7$, and $|\theta| = 26°$ for $\omega/N = 0.9$, in close accordance with the condition 1.44). Comparison between measured and predicted angles is plotted in Figure (3) for a wide range of ω/N To under the physics better we note first that the phase velocity is

$$\vec{C} = \pm \frac{\omega}{k^2} \, (\alpha, \beta) \tag{1.45}$$

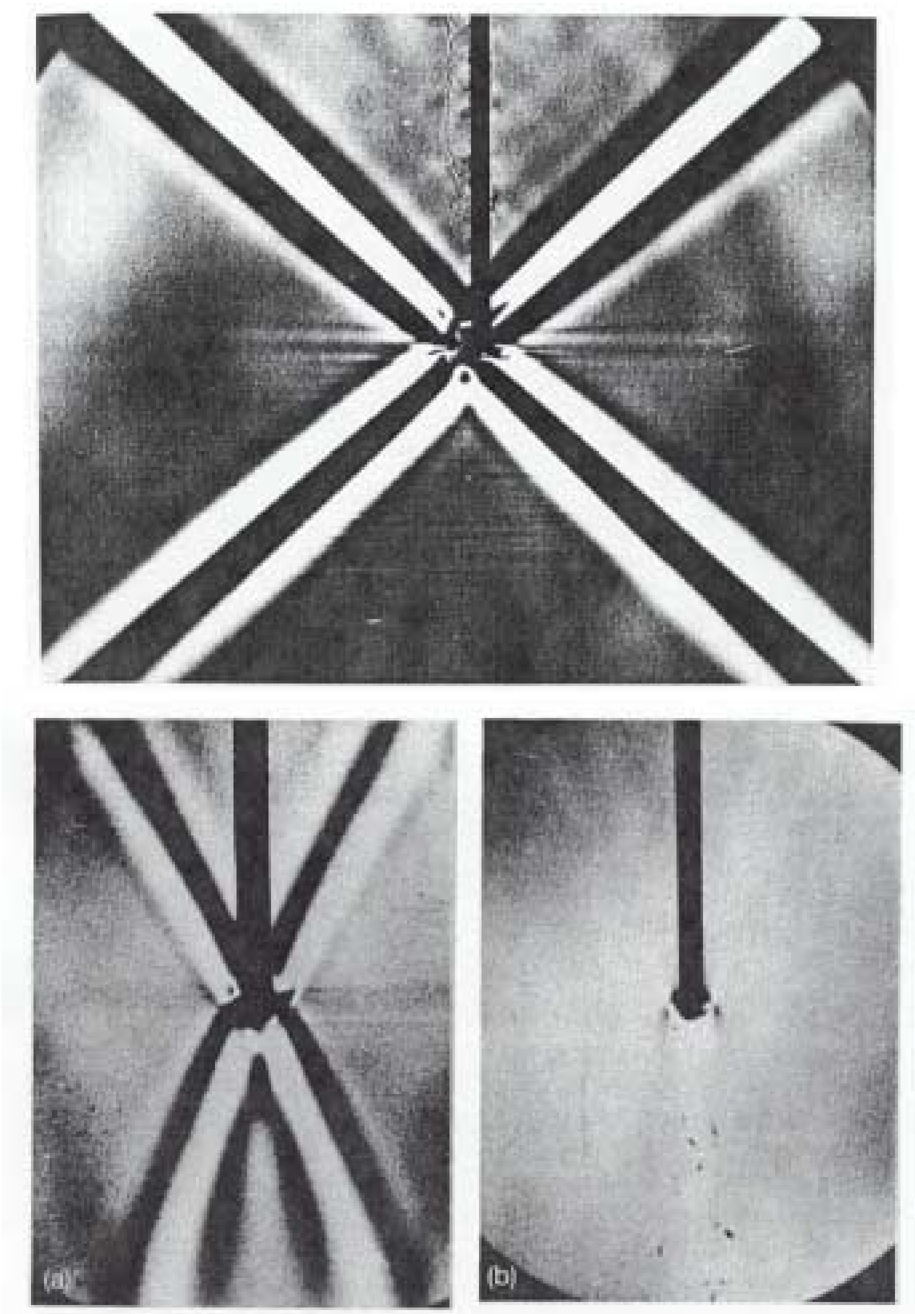

Figure 2: St Andrew's Cross in a stratified fluid

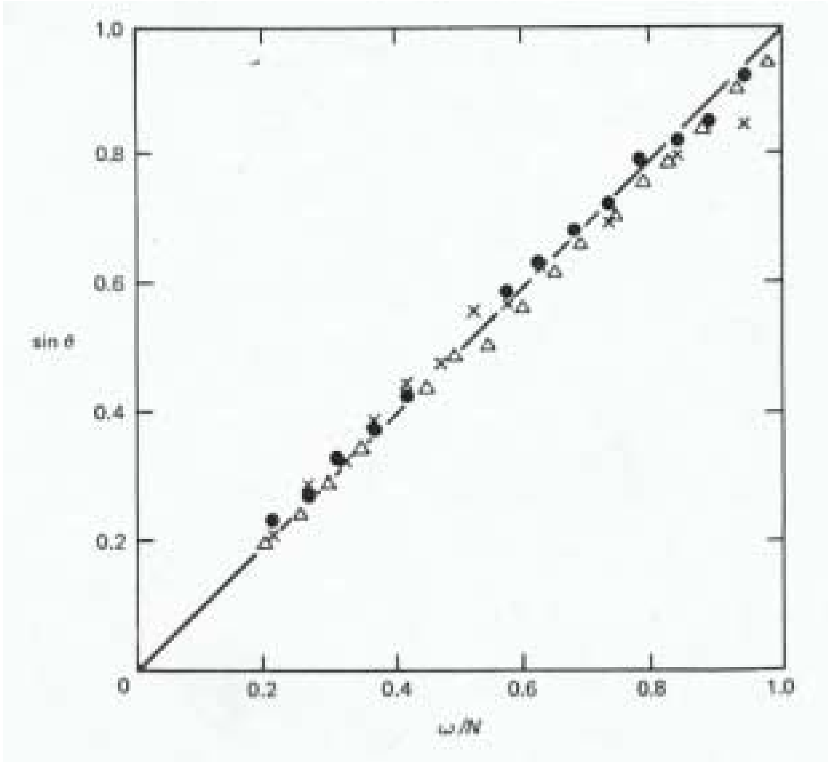

Figure 3: Comparison of measured and predicted angles of internal waves

while the group velocity components are

$$\begin{aligned} C_{gx} &= \frac{\partial \omega}{\partial \alpha} = \pm N \left(\frac{1}{k} - \frac{\alpha}{k^2} \frac{\alpha}{k} \right) \\ &= \pm \frac{N}{k} \left(1 - \frac{\alpha^2}{k^2} \right) = \pm \frac{N}{k^3} \beta^2 \end{aligned}$$

$$C_{gz} = \frac{\partial \omega}{\partial \beta} = \mp \frac{\alpha \beta}{k^3}. \tag{1.46}$$

Thus

$$\vec{C_g} = \pm N \frac{\beta}{k^2} \left(\frac{\beta}{k}, \frac{-\alpha}{k} \right). \tag{1.47}$$

Therefore, the group velocity is perpendicular to the phase velocity,

$$\vec{C_g} \cdot \vec{C} = 0. \tag{1.48}$$

Since

$$\vec{C} + \vec{C_g} = \pm \frac{N}{k^3} \left(\alpha^2 + \beta^2, 0 \right) = \pm \frac{N}{k^2} (k, 0) \tag{1.49}$$

the sum of $\vec{C}$ and $\vec{C_g}$ is a horizontal vector, as shown by any of the sketches in Figure 3. Note that when tkhe phase velocity as an upward component, the group velocity has a downward component, and vice versa. Now let us consider energy transport. from

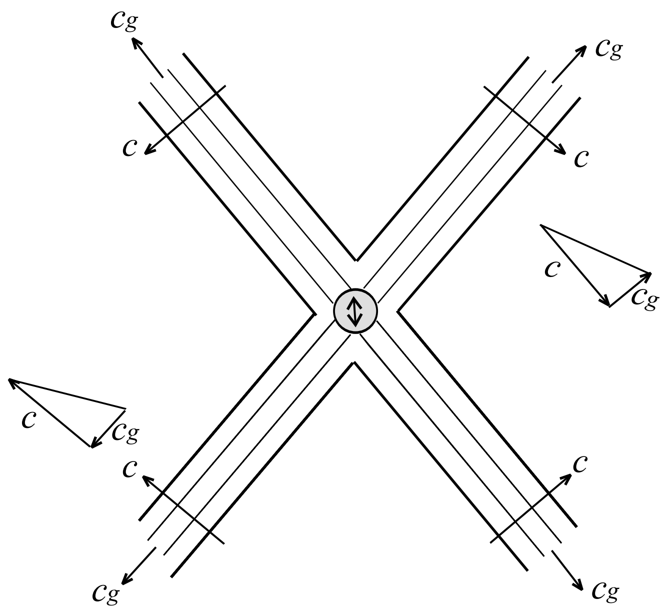

Figure 4: Phase and group velocities

(1.10) we get

$$-p'_x = \overline{\rho}\psi_{zt} = \overline{\rho}\omega\beta\psi_o e^{i(\alpha x+\beta z-\omega t)}$$

hence the dynamic pressure is

$$p' = i\omega\overline{\rho}\frac{\beta}{k}\psi_o e^{i(\alpha x+\beta z-\omega t)} \tag{1.50}$$

The fliud velocity is easily calculated

$$\vec{q}' = (u', v') = (\psi_z, -\psi_x) = i\overline{\rho}(\beta, -\alpha)\psi_o e^{i(\alpha x+\beta z-\omega t)} \tag{1.51}$$

The averaged rate of energy transport is therefore

$$\vec{E} = \frac{1}{2}\overline{\rho}^2|\psi|^2\frac{\beta}{\alpha}\,(\beta, \alpha) \tag{1.52}$$

which is in the same direction of the group velocity.

Now returnning to the St. Andrews cross in figure (2). Energy must radiate outward from the oscillating source, hence the group velocity vectors must all be outward. The crests in the beam in the first quadrant must be in the south-easterly direction. Simlarly the crests in all four beams must be outward and toward the horizontal axis. Movie records indeed confirm this prediction. Within each of the four beams which have widths comparable to the cylinder diameter, only one or two wave lengths can be seen.

For another interesting feature, consider the reflection of an internal wave from a slope.

Recall that $\theta = \pm \cos^{-1} \frac{\omega}{N}$, i.e., for a fixed frequency there are only two allowable directions with respect to the horizon. Relative to the sloping bottom inclined at θ_o the inclinations of the incident and reflected waves must be different, and are respectively $\theta + \theta_o$ and $\theta - \theta_o$, see figure.

Let ξ be along, and η be normal to the slope. Since the slope must be a streamline, $\psi_i + \psi_r$ must vanish along $\eta = 0$ and be proportional to $e^{i(\alpha\xi - \omega t)}$; the total stream function must be of the form

$$\psi_i \, e^{i(k_t^{(i)}\xi - \omega t)} + \psi_r \, e^{i(k_t^{(r)}\xi - \omega t)} \propto \sin \beta\eta e^{i(\alpha\xi - \omega t)}.$$

In particular the wavenumber component along the slope must be equal,

$$k_t^{(i)} = k_t^{(r)} = \alpha$$

Therefore

$$k^{(i)} \cos(\theta + \theta_o) = k^{(r)} \cos(\theta - \theta_o),$$

which implies that

$$k^{(i)} \neq k^{(r)}. \tag{1.53}$$

as sketched in Figure 5. The incident wave and the reflected wave have different wavelengths! If $\theta < \theta_o$, there is no reflection; refraction takes place instead.

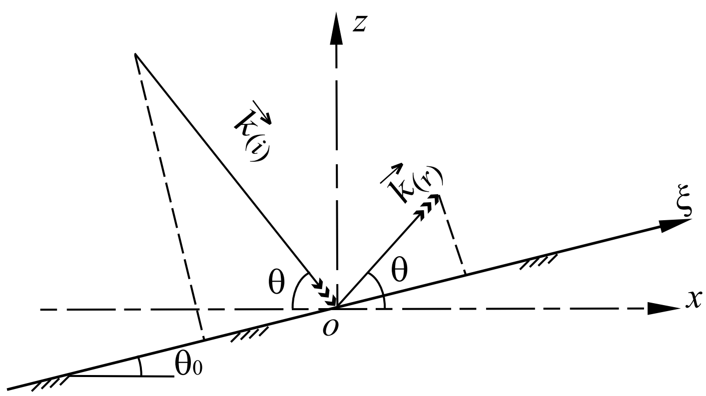

Figure 5: Internal wave reflected by in inclined surface