I-campus project

School-wide Program on Fluid Mechanics

Modules on Waves in fluids

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CHAPTER FIVE

REFLECTION, TRANSMISSION, AND DIFFRACTION

Scope:

Reflection of sound at an interface. (Reference : Brekhovskikh and Godin §.2.2.)

Diffraction by a circular cylinder, theory and simulation.

Diffraction by a wedge

- Parabolic approximation

- Exact theory and Numerical simulation.

1 Introduction to two dimensional scattering

When waves are intercepted by a physical boundary, reflection and scattering occur. Since in principle any transient signal can be represented as a Fourier integral of simple harmonic waves within a wide specrum of frequencies, it is a basic problem to study scattering of monochromatic waves.

For sound in a homogeneous fluid, the veloctiy potential defined by $\mathbf{u} = \nabla\phi$ satisfies

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \tag{1.1}$$

where c denotes the sound speed. Recall that the fluid pressure $p = -\rho_o \partial\phi/\partial t$ also satisfies the same equation.

We first generalize the plane sinusoidal wave in three dimensional space

$$\phi(\mathbf{x}, t) = \phi_o e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \phi_o e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)} \tag{1.2}$$

where $\mathbf{n}$ is the unit vector in the direction of $\mathbf{k}$. Here the phase function is

$$\theta(\mathbf{x}, t) = \mathbf{k}\cdot\mathbf{x} - \omega t \tag{1.3}$$

The equation of constant phase $\theta(\mathbf{x}, t) = \theta_o$ describes a moving surface. The wave number vector $\mathbf{k} = k\mathbf{n}$ is defined to be

$$\mathbf{k} = k\mathbf{n} = \nabla\theta \tag{1.4}$$

hence is orthogonal to the surface of constant phase, and represens the direction of wave propagation. The frequency is defined to be

$$\omega = -\frac{\partial\theta}{\partial t} \tag{1.5}$$

Is (1.2) a solution? Let us check (2.1).

$$\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi = i\mathbf{k}\phi$$

$$\nabla^2\phi = \nabla \cdot \nabla\phi = i\mathbf{k} \cdot i\mathbf{k}\phi = -k^2\phi$$

$$\frac{\partial^2\phi}{\partial t^2} = -\omega^2\phi$$

Hence (1.1) is satisfied if

$$\omega = kc \tag{1.6}$$

Scattering by an object has been the focus of research in physics, electrical, acoustical and oceanographical engineering for a long time. Depending on the geometry, the mathematics can be quite involved. Mountains of literatures on analytical and numerical methods can be found. In this chapter we shall limit our study to two space dimensions. For a plane sound wave of single frequency scattered by a cylinder whose axis is parallel to the incident crests, the two-dimensional, time-dependent potential can be written as

$$\Phi(x, y, t) = \Re\left[\phi(x, y)e^{-i\omega t}\right] \tag{1.7}$$

where the potential amplitude ϕ is governed by the Helmholtz equation

$$\nabla^2\phi + k^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi^2}{\partial y^2} + k^2\phi = 0, \quad k = \frac{\omega}{c} \tag{1.8}$$

On the rigid and perfectly reglective boundary B the normal velocity vanishes,

$$\frac{\partial\phi}{\partial n} = 0 \tag{1.9}$$

Let the total wave be the sum of the incident and scattered waves

$$\phi = \phi_I + \phi_S \tag{1.10}$$

then the scattered waves must further satisfy the *radiation condition* at infinity, i.e., it can only radiate energy outward from the scatterer.

Th preceding boundary value problem goeverns wave scattering in a variety of physical contexts. Elastic shear waves scattered by a cylinderical cavity waves is one example. The scattering of surface water waves in a sea of constant depth is in principle thee dimensional, yet it can be reduced to the same two-dimensional boundary value problem above, if the scatterer (a breakwater, a storage tank , etc.,) has vertical side walls extending the entire water depth. Let us explain why.

Conder a vertical cylindrical structure of of arbitrary plan form in a sea of constant depth h. A train of monochromatic waves is incident from infinity at the angle α with respect the x axis. The still water surface is in the x, y plane.

In the water region defined by $0 \geq z \geq -h$, the velocity potential $\Phi(r, \theta, z, t)$ must satisfy the Laplace equation,

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{1.11}$$

and subject to the linearized free surface boundary conditions

$$\frac{\partial \Phi}{\partial t} = -g\zeta \tag{1.12}$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \tag{1.13}$$

which can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0, \quad z = 0. \tag{1.14}$$

Along the impermeable bottom and coasts, the no flux boundary conditions are

$$\frac{\partial \Phi}{\partial z} = 0 \qquad on \;\; z = -h \tag{1.15}$$

$$\frac{\partial \Phi}{\partial \theta} = 0 \qquad at \;\; \theta = 0 \;\; and \;\; \nu\pi \tag{1.16}$$

The incident wave train is given by

$$\Phi_i = \frac{-igA_0}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \phi(r,\theta) e^{-ikr\cos(\theta-\alpha)-i\omega t} \tag{1.17}$$

where k is the real wavenumber satisfying the dispersion relation

$$\omega^2 = gk \tanh kh, \tag{1.18}$$

and $\pi + \alpha$ is the angle of incidence measured from the x axis. A_0 is the incident wave amplitude.

Because of the vertical side-walls, we assume

$$\Phi(r,\theta,z,t) = A_0 \phi(x,y,t) \frac{\cosh k(z+h)}{\cosh kh} e^{-i\omega t} \tag{1.19}$$

where $\phi(x,y,t)$ is the horizontal pattern of the potential normlized for an incident wave of unit amplitude, and is related to the amplitude of the free surface displacement $\eta(x,y)$ by

$$\phi(x,y) = -\frac{ig\eta(x,y)}{\omega} \tag{1.20}$$

Substituting (1.19) into the Laplace equation and using both the kinematic and dynamic boundary conditions on the free surface, the Laplace equation in (x,y,z) is then reduced to the two dimensional Helmholtz equation in (x,y),

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0, \quad (x,y) \text{ in the fluid.} \tag{1.21}$$

The no normal flux boundary condition on the rigid vertical wall B becomes

$$\mathbf{n} \cdot \nabla \eta = \frac{\partial \phi}{\partial n} = 0, \quad (x,y) \in B \tag{1.22}$$

since the normal to the cylinder wall is horizontal. Therefore the three-dimensional water-wave problem is mathematically equivalent to the two-dimension sound problem,

2 Sound reflection and transmission across an interface

Consider two semi-infinite fluids separated by the plane interface along $z = 0$. The sound speeds in the upper and lower fluids are c and c_1 respectively. Let a plane incident wave arive from $z > 0$ at the incident angle of θ with respect to the z axis,

$$p_i = \exp[ik(x \sin\theta - z\cos\theta)] \tag{2.1}$$

implying that

$$\mathbf{k}^i = (k_x^i, k_z^i) = k(\sin\theta, -\cos\theta) \tag{2.2}$$

The motion is confined in the x, z plane.

On the same (incidence) side of the interface we have the reflected wave

$$p_r = R\exp[ik(x\sin\theta + z\cos\theta)] \tag{2.3}$$

where R denotes the reflection coefficient. The wavenumber vector is

$$\mathbf{k}^r = (k_x^r, k_z^r) = k(\sin\theta, \cos\theta) \tag{2.4}$$

In the lower medium $z < 0$ the transmitted wave has the pressure

$$p_t = T\exp[ik_1(x\sin\theta_1 - z\cos\theta_1)] \tag{2.5}$$

where T is the transmission coefficient. Along the interface $z = 0$ we require the continutiy of pressure and normal velocity, i.e.,

$$[p] = 0, \quad z = 0 \tag{2.6}$$

and

$$[w] = 0 \quad z = 0, \tag{2.7}$$

where the square brackets signify the jump across the interface:

$$[f] \equiv f(z = 0+) - f(z = 0-) \tag{2.8}$$

We define the impedance of a simple harmonic waves by

$$Z = -\frac{p}{w} \tag{2.9}$$

where w is the vertical component of the fluid velocity. Because

$$\rho\frac{\partial w}{\partial t} = -i\omega\rho w = -\frac{\partial p}{\partial z}, \tag{2.10}$$

$$\frac{p}{w} = -\frac{-i\omega\rho p}{\frac{\partial p}{\partial z}} \tag{2.11}$$

It follows from the two continuity requirements that the impedance must be continuous

$$[Z] = 0 \quad z = 0 \tag{2.12}$$

Note first that to satisfy the conditions of continuity for all x it is necessary that the y factors match, so that

$$k \sin\theta = k_1 \sin\theta_1 \tag{2.13}$$

or

$$\frac{\sin\theta}{c} = \frac{\sin\theta_1}{c_1} \tag{2.14}$$

Eq. (2.13) or (2.14) is the famous Snell's law of refraction. If $c_1 < c$, waves are incident from the faster medium, the direction of the refracted (or transmitted) wave is closer to the normal to the interface. Now (2.6) requires that

$$1 + R = T \tag{2.15}$$

The impedance of the incident wave, the reflected wave, and the transmitted waves are respectively

$$Z_i = \frac{\rho c}{\cos\theta}, \quad Z_r = -\frac{\rho c}{\cos\theta}, \quad Z_1 = \frac{\rho_1 c_1}{\cos\theta_1} \tag{2.16}$$

which are all constants, and the total impedance on the incidence/reflection side is

$$Z = \frac{\rho c}{\cos\theta} \frac{\exp(-2ikz\cos\theta) + R}{\exp(-2ikz\cos\theta) - R} \tag{2.17}$$

which is in general a complex function of z. Next we impose (2.6) and get

$$Z_1 = \frac{\rho c}{\cos\theta} \frac{1+R}{1-R} \tag{2.18}$$

hence

$$R = \frac{Z_1 \cos\theta - \rho c}{Z_1 \cos\theta + \rho c} \tag{2.19}$$

This formula is written in a general form where the impedance of the lower medium can be anything . For the present example it is given by (2.16) and

$$R = \frac{\rho_1 c_1 \cos\theta - \rho c \cos\theta_1}{\rho_1 c_1 \cos\theta + \rho c \cos\theta_1} \tag{2.20}$$

Let

$$m = \frac{\rho_1}{\rho}, \quad n = \frac{c}{c_1} \tag{2.21}$$

where the ratio of sound speeds n is called the index of refraction. We get after using Snell's law that

$$R = \frac{m\cos\theta - n\cos\theta_1}{m\cos\theta + n\cos\theta_1} = \frac{m\cos\theta - n\sqrt{1 - \frac{\sin^2\theta}{n^2}}}{m\cos\theta + n\sqrt{1 - \frac{\sin^2\theta}{n^2}}} \tag{2.22}$$

The transmission coefficient follows readily from (2.15),

$$T = 1 + R = \frac{2m\cos\theta}{m\cos\theta + n\sqrt{1 - \frac{\sin^2\theta}{n^2}}} \tag{2.23}$$

We now examine the physics.

If $n = c/c_1 > 1$, the incidence is from a faster to a slower medium, then R is always real. If however $n < 1$ then $\theta_1 > \theta$. There is a critical incidence angle θ_c, defined by

$$\sin\theta_c = n \tag{2.24}$$

beyond which ($\theta > \theta_c$) the square roots above become imaginary. We must then take

$$\cos\theta_1 = \sqrt{1 - \frac{\sin^2\theta}{n^2}} = i\sqrt{\frac{\sin^2\theta}{n^2} - 1} \tag{2.25}$$

This means that the reflection coefficient is now complex

$$R = \frac{m\cos\theta - in\sqrt{\frac{\sin^2\theta}{n^2} - 1}}{m\cos\theta + in\sqrt{\frac{\sin^2\theta}{n^2} - 1}} \tag{2.26}$$

with $|R| = 1$, implying complete reflection. As a check the transmitted wave is now given by

$$p_t = T\exp\left[k_1\left(ix\sin\theta_1 + z\sqrt{\sin^2\theta/n^2 - 1}\right)\right] \tag{2.27}$$

so the amplitude attenuates exponentially in z as $z \to -\infty$. Thus the wave train cannot penetrate much below the interface.

The dependence of R on various parameters is best displayed in the complex plane $R = \Re R + i\Im R$.

Case 1: $n > 1$. Here R is always real.

For normal incidence $\theta = 0$,

$$R = \frac{m-n}{m+n} \tag{2.28}$$

$R > 0$ if $n < m$ and $R < 0$ if $n > m$. In either case $|R| < 1$ For glancing incidence $\theta = \pi/2$, $R = -1$. For any intermediate incidence angles, R falls in the segment of the real axis as shown in figure 1.a. and 1.b.

Case 2. $n < 1$ then R is real only if $\theta < \theta_c$, otherwise R becomes complex and has the unit amplitude. It is clear from (2.26) that $\Im R < 0$ so that R falls on the half circle in the lower half of the complex plane as shown in figure 1.c and 1.d.

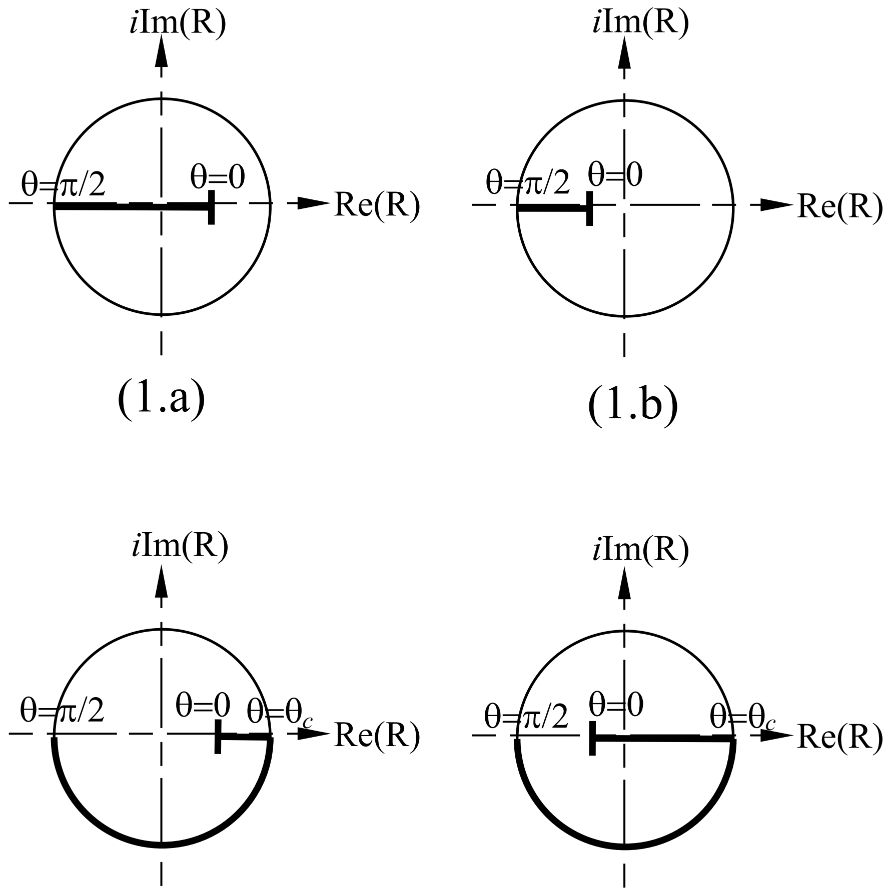


Figure 1: Complex reflection coefficient

3 Scattering by a circular cylnder

3.1 Solution in polar coordinates

We study a cylindrical scatter of circular cross section with radius a. The boundary condition on the cylinder surface is

$$\frac{\partial \phi}{\partial r} = 0, \quad r = a \tag{3.1}$$

It is convenient to employ polar coordinates r, θ where

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.2}$$

The governing equation then reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + k^2\phi = 0 \tag{3.3}$$

Since ϕ_I satisfies the preceding equation, so does ϕ_S.

Let the incident wave ϕ_I be a plane wave inclined at the angle of incidence θ_o with respect to the positive x axis. In polar coordinates we write

$$\mathbf{k} = k(\cos\theta_o, \sin\theta_o), \quad \mathbf{x} = r(\cos\theta, \sin\theta) \tag{3.4}$$

$$\phi_I = A\exp\left[ikr(\cos\theta_o\cos\theta + \sin\theta_o\sin\theta)\right] = Ae^{ikr\cos(\theta-\theta_o)} \tag{3.5}$$

It can be shown (see Appendix A) that the plane wave can be expanded in Fourier-Bessel series :

$$e^{ikr\cos(\theta-\theta_o)} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr)\cos n(\theta-\theta_o) \tag{3.6}$$

where ϵ_n is the Jacobi symbol:

$$\epsilon_0 = 0, \quad \epsilon_n = 2,\ n = 1, 2, 3, \ldots \tag{3.7}$$

Each term in the series (3.6) is called a partial wave.

By the method of separation of variables,

$$\phi_S(r,\theta) = R(r)\Theta(\theta)$$

we find

$$r^2R'' + rR' + (k^2r^2 - n^2)R = 0, \quad \text{and} \quad \Theta'' + n^2\Theta = 0$$

where $n = 0, 1, 2, \ldots$ are eigenvalues in order that Θ is periodic in θ with period 2π. For each eigenvalue n the possible solutions are

$$\Theta_n = (\sin n\theta, \cos n\theta),$$

$$R_n = \left(H_n^{(1)}(kr), H_n^{(2)}(kr)\right),$$

where $H_n^{(1)}(kr), H_n^{(2)}(kr)$ are Hankel functions of the first and second kind, related to the Bessel and Weber functions by

$$H_n^{(1)}(kr) = J_n(kr) + iY_n(kr), \quad H_n^{(2)}(kr) = J_n(kr) - iY_n(kr) \tag{3.8}$$

The most general solution to the Helmholtz equation is

$$\phi_S = A\sum_{n=0}^{\infty}(A_n\sin n\theta + B_n\cos n\theta)\left[C_nH_n^{(1)}(kr) + D_nH_n^{(2)}(kr)\right], \tag{3.9}$$

For large radius the asymptotic form of the Hankel functions behave as

$$H_n^{(1)} \sim \sqrt{\frac{2}{\pi kr}} e^{i(kr - \frac{\pi}{4} - \frac{n\pi}{2})}, \quad H_n^{(2)} \sim \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \frac{\pi}{4} - \frac{n\pi}{2})} \tag{3.10}$$

In conjunction with the time factor $\exp(-i\omega t)$, $H_n^{(1)}$ gives outgoing wavew while $H_n^{(2)}$ give incoming waves. To satisfy the radiation condition, we must discard all terms involving $H_n^{(2)}$. From here on we shall abbreviate $H_n^{(1)}$ simply by H_n. The scattered wave is now

$$\phi_S = A \sum_{n=0}^{\infty} (A_n \sin n\theta + B_n \cos n\theta) H_n(kr) \tag{3.11}$$

The expansion coefficients (A_n, B_n) must be chosen to satisfy the boundary condition on the cylinder surface. Without loss of generality we can take $\theta_o = 0$. On the surface of the cylindrical cavity $r = a$, we impose

$$\frac{\partial \phi_I}{\partial r} + \frac{\partial \phi_S}{\partial r} = 0, \quad r = a$$

It follows that $A_n = 0$ and

$$\epsilon^n i^n A J_n'(ka) + B_n k H_n'(ka) = 0, \quad n = 0, 1, 2, 3, \dots n$$

where prinmes denote differentiation with respect to the argument. Hence

$$B_n = -A\epsilon_n i^n \frac{J_n'(ka)}{H_n'(ka)}$$

The sum of incident and scattered waves is

$$\phi = A \sum_{n=0}^{\infty} e_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right] \cos n\theta \tag{3.12}$$

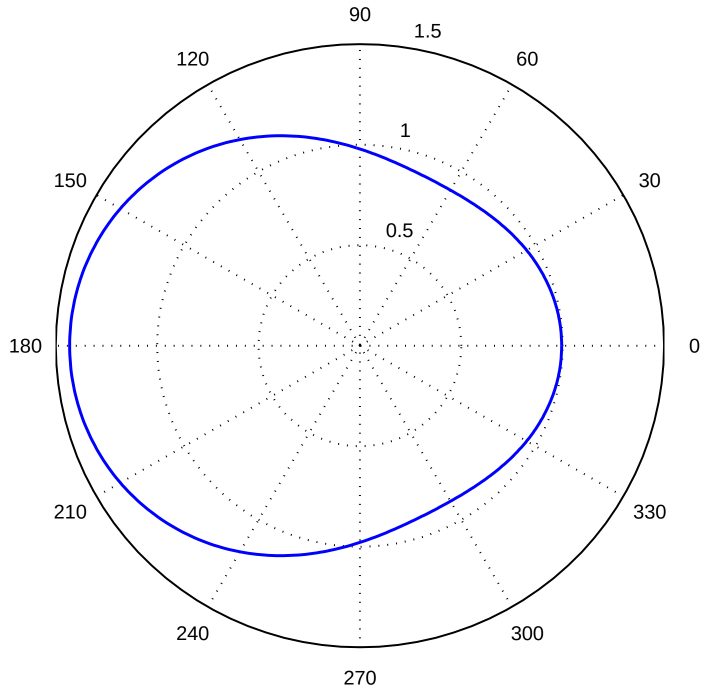
and

$$\Phi = Ae^{-i\omega t} \sum_{n=0}^{\infty} e_n i^n \left[J_n(kr) - \frac{J_n'(ka)}{H_n'(ka)} H_n(kr) \right] \cos n\theta \tag{3.13}$$

The numerical simulations can be seen for a wide range of ka on the web (give web link).

Of practical interest is the angular variation of pressure on the surface of the cylinder $r = a$. Figure 2 shows that for small ka the pressure is relatively uniform in all directions. For increasingly large ka, waves become stronger on the reflection side (reaching 2 at the back $\theta = \pi$). On the shadow side the wave intensity is weak.

(a) ka= 0.5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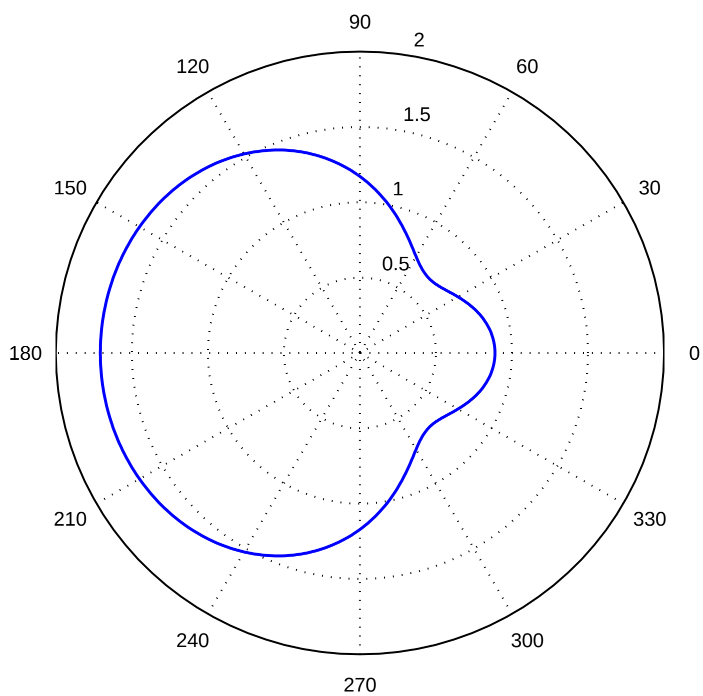
(b) ka= 1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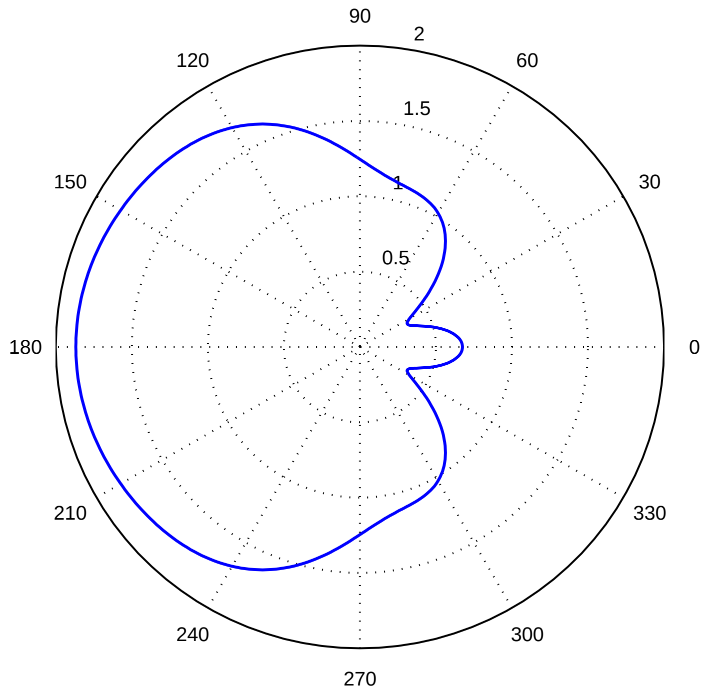
(c) ka= 2.5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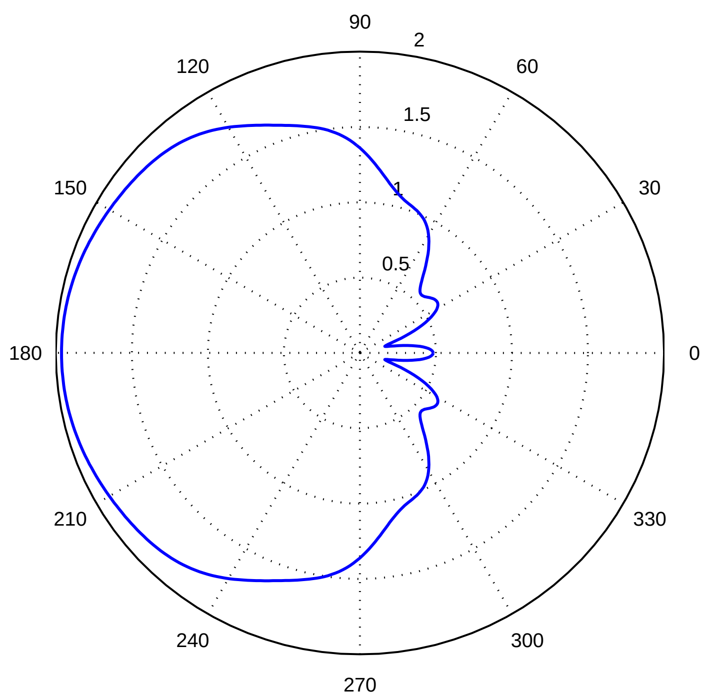
(d) ka= 5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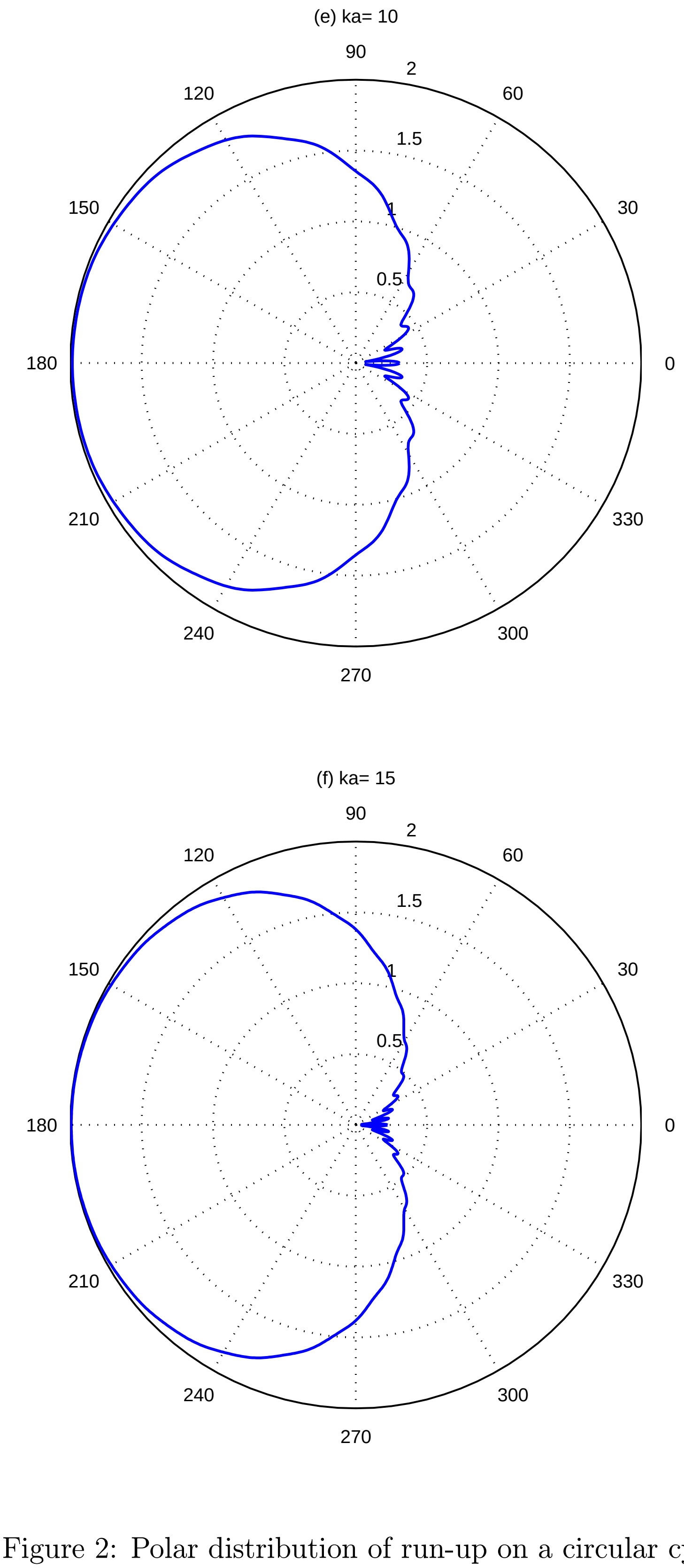


Figure 2: Polar distribution of run-up on a circular cylinder

It is also interesting to examine certain limits. For very long long waves $ka \ll 1$ the expansions for Bessel functions for small argument may be used,

$$J_n(x) \sim \frac{x^n}{2^n n!}, \quad Y_n(x) \sim \frac{2}{\pi}\log x, \quad Y_n(x) \sim \frac{2^n(n-1)!}{\pi x^n} \tag{3.14}$$

Then the scattered wave has the potential

$$\begin{aligned}\frac{\phi_S}{A} &\sim -H_0(kr)\frac{J"_0)(ka)}{H_0')(ka)} - 2iH_1(kr)\frac{J"_1)(ka)}{H_1')(ka)}\cos\theta + O(ka)^3 \\ &= \frac{\pi}{2}(ka)^2\left(-\frac{i}{2}H_0(kr) - H_1(kr)\cos\theta\right) + O(ka)^3 \end{aligned} \tag{3.15}$$

The term $H_0(kr)$ coresponds to an oscillating source which sends istropic waves in all directions. The second term is a dipole sending scattered waves mostly in forward and backward directions.

For large kr, the angular variation can be obtained by using the asymptotic formulas to get

$$\phi_S \sim A\sum_{n=0}^{\infty}\left(A_n \sin n\theta + B_n \cos n\theta\right) e^{-in\pi/2}\sqrt{\frac{2}{\pi kr}}e^{ikr - i\pi/4} \tag{3.16}$$

Let us define the dimensionless directivity factor

$$\mathcal{A}(\theta) = \sum_{n=0}^{\infty}\left(A_n \sin n\theta + B_n \cos n\theta\right) e^{-in\pi/2} \tag{3.17}$$

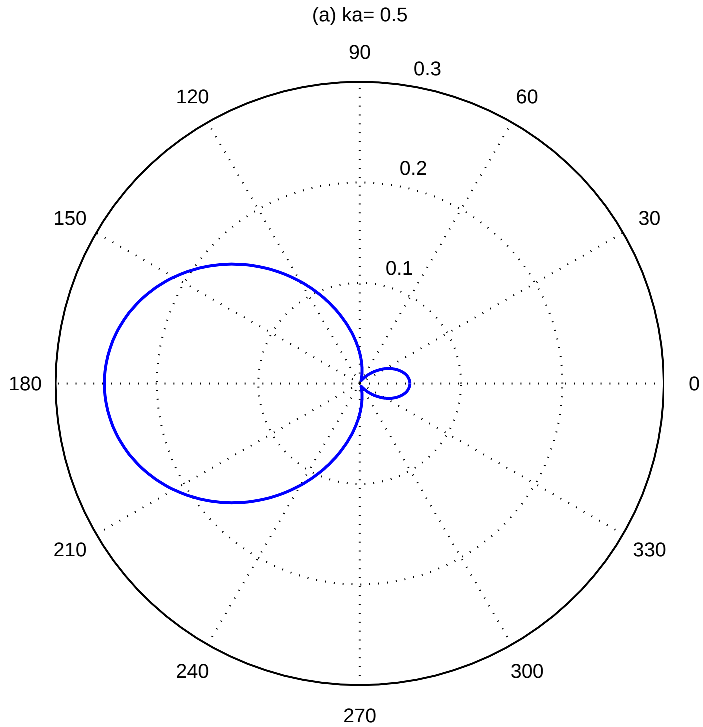
which indicates the angular variation of the far-field amplitude, then

$$\phi_S \sim A\mathcal{A}(\theta)\sqrt{\frac{2}{\pi kr}}e^{ikr - i\pi/4} \tag{3.18}$$

This expression exhibits clearly the asymptotic behaviour of ϕ_S as an outgoing wave. By differentiation, we readily see that

$$\lim_{kr\to\infty}\sqrt{r}\left(\frac{\partial \phi_S}{\partial r} - \phi_S\right) = 0 \tag{3.19}$$

which is one way of stating the radiation condition for two dimensional scattered waves. The far field pattern of $|\mathcal{A}(\theta)|^2$ for various ka can be numerically computed as shown in fig (3.1).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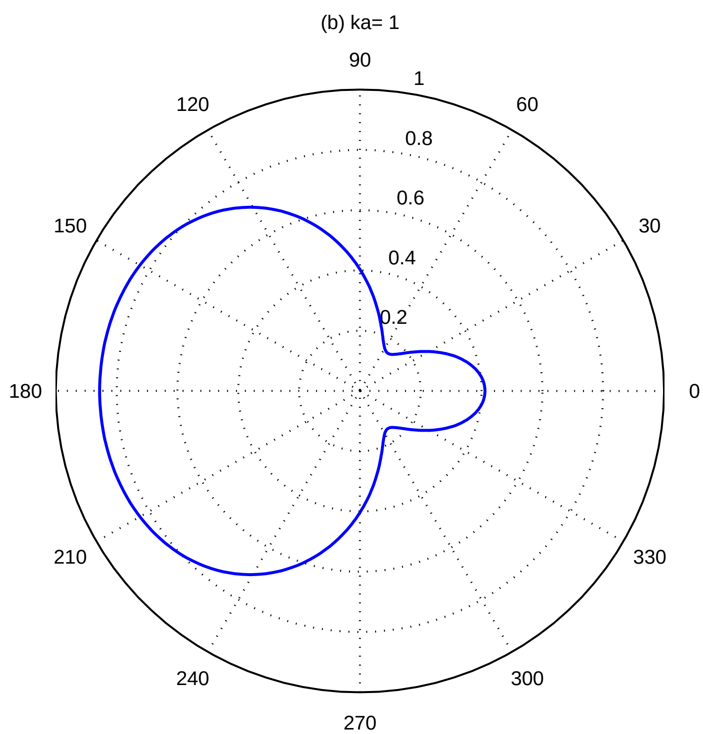
180
0
210
330
240
300
270


(b) ka= 1
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1
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0
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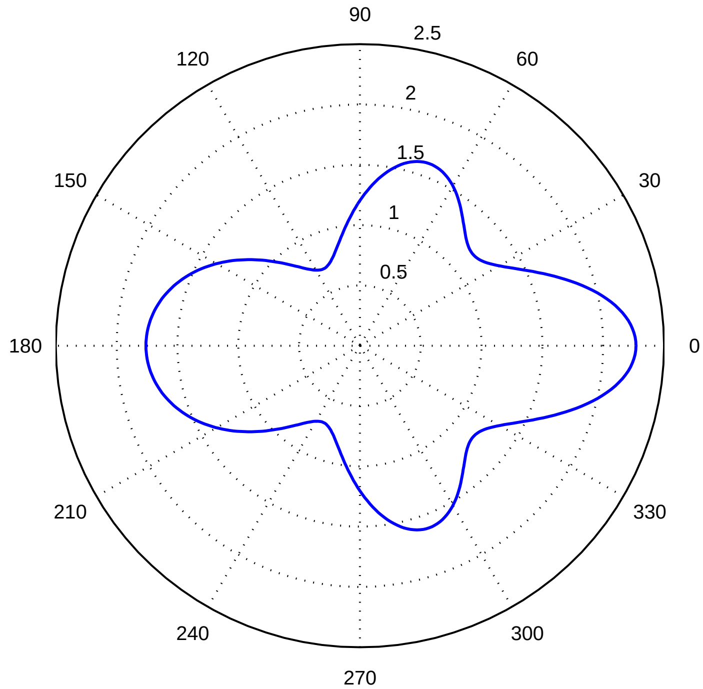
240
300
270

(c) ka= 2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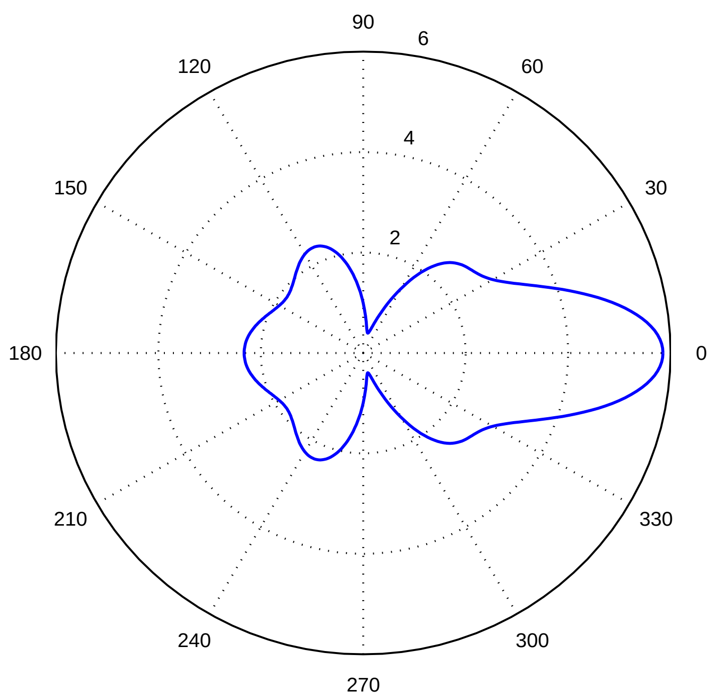
(d) ka= 3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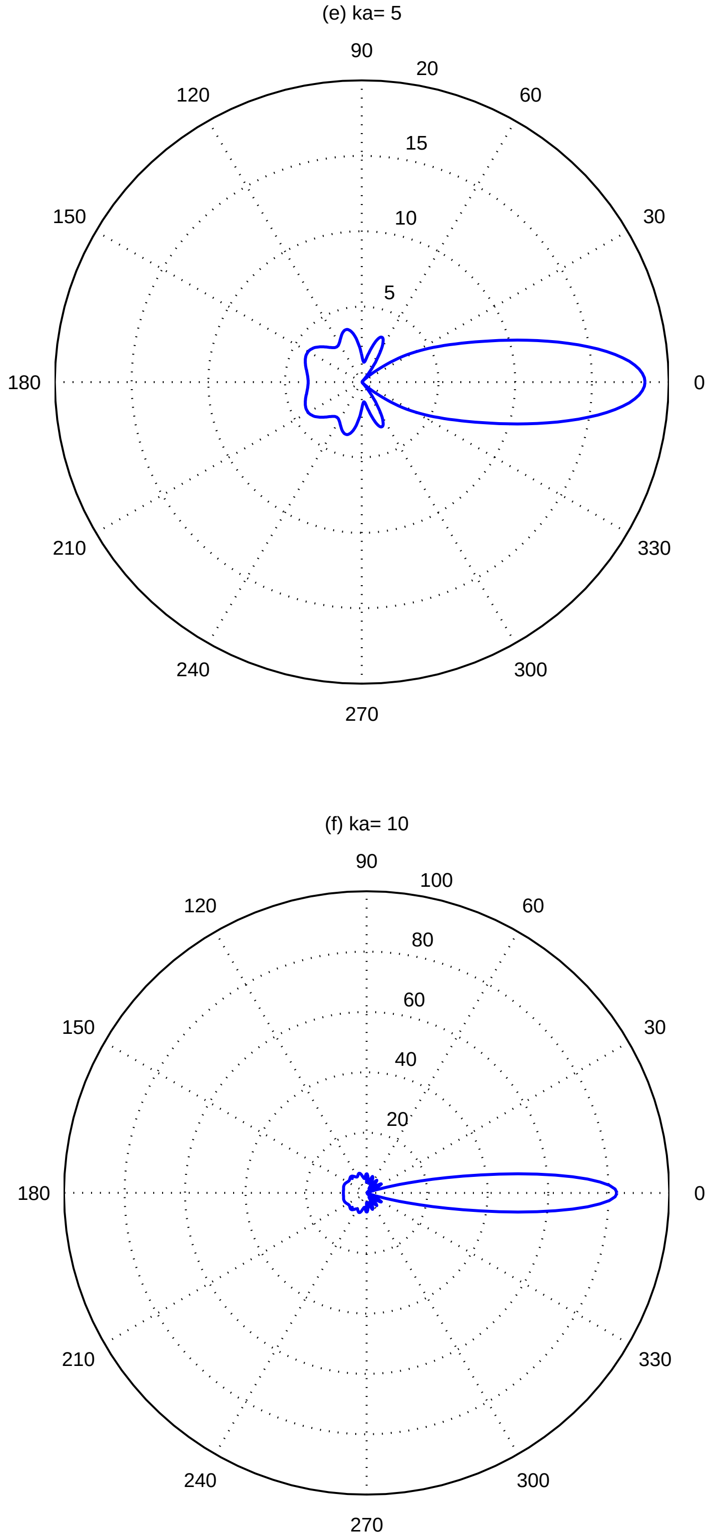

Figure 3: Far-field energy intensity as a function of direction

4 Energy conservation and a general theorem

At any radius r the sound pressure and radial fluid velocity are respectively,

$$p = -\rho_o \frac{\partial \phi}{\partial t}, \quad \text{and} \quad = \frac{\partial \phi}{\partial r} \tag{4.1}$$

The total rate of energy outflux by the scattered wave is

$$\begin{aligned} r \int_0^{2\pi} d\theta \overline{\tau_{rz} \frac{\partial u_z}{\partial t}} &= r \int_0^{2\pi} d\theta \overline{\Re \left[-k^2 \frac{\partial \phi}{\partial r} e^{-i\omega t} \right] \Re \left[i\omega k^2 \phi e^{-i\omega t} \right]} \\ &= -\frac{\mu \omega k^4 r}{2} \int_0^{2\pi} d\theta \Re \left[i\phi^* \frac{\partial \phi}{\partial r} \right] = -\frac{\mu \omega k^4 r}{2} \Im \int_0^{2\pi} d\theta \left[\phi^* \frac{\partial \phi}{\partial r} \right] \end{aligned} \tag{4.2}$$

where overline indicates time averaging over a wave period $2\pi/\omega$.

The energy scattering rate is therefore

$$r \int_0^{\infty} d\theta \overline{p u_r} = \frac{\omega \rho_o r}{2} \Re \int_C d\theta \left(-i\phi^* \frac{\partial \phi}{\partial r} \right) = -\frac{\omega \rho_o r}{2} \Im \int_C d\theta \left(\phi^* \frac{\partial \phi}{\partial r} \right) \tag{4.3}$$

We now derive a general theorm for this quantity.

For the same scatterer and the same frequency ω, different angles of incidence θ_j define different scattering problems ϕ_j. In particular at infinty, we have

$$\phi_j \sim A_j \left\{ e^{ikr\cos(\theta - \theta_j)} + \mathcal{A}_j(\theta) \sqrt{\frac{2}{\pi k r}} e^{ikr - i\pi/4} \right\} \tag{4.4}$$

Let us apply Green's formula to ϕ_1 and ϕ_2 over a closed area bounded by a closed contour C,

$$\iint_S \left(\phi_2 \nabla^2 \phi_1 - \phi_1 \nabla^2 \phi_2 \right) dA = \int_B \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n} \right) ds + \int_C ds \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_1}{\partial n} \right) ds$$

where $\mathbf{n}$ refers to the unit normal vector pointing out of S. The surface integral vanishes on account of the Helmholtz equation, while the line integral along the cavity surface vanishes by virture of the boundary condition, hence

$$\int_C ds \left(\phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n} \right) ds = 0 \tag{4.5}$$

By similar reasioning, we get

$$\int_C ds \left(\phi_2 \frac{\partial \phi_1^*}{\partial n} - \phi_1^* \frac{\partial \phi_2}{\partial n} \right) ds = 0 \tag{4.6}$$

where ϕ_1^* denotes the complex conjugate of ϕ_1.

Let us choose $\phi_1 = \phi_2 = \phi_o$ in (4.6), and get

$$\int_C ds \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n} \right) ds = 2\Im \left(\int_C ds\, \phi \frac{\partial \phi^*}{\partial n} \right) = 0 \tag{4.7}$$

This mathematical result implies the conservation of energy. Physically, across any circle the net rate of energy flux vanishes, i.e., the scattered power must be balanced by the incident power.

Making use of (4.4) we get

$$\begin{aligned}
0 = \Im \int_0^{2\pi} r d\theta & \left[e^{ikr\cos(\theta-\theta_o)} + \sqrt{\frac{2}{\pi kr}} \mathcal{A}_o(\theta) e^{ikr - i\pi/4} \right] \\
& \cdot \left[-ik\cos(\theta-\theta_o) e^{ikr\cos(\theta-\theta_o)} - ik\sqrt{\frac{2}{\pi kr}} \mathcal{A}_o^*(\theta) e^{ikr - i\pi/4} \right] \\
& = \Im \int_0^{2\pi} r d\theta \Big\{ -ik\cos(\theta-\theta_o) + \frac{2}{\pi kr}(-ik)|\mathcal{A}_o|^2 \\
& + e^{ikr[\cos\theta-\theta_o)-1]+i\pi/4}(-ik)\sqrt{\frac{2}{\pi kr}}\mathcal{A}_o^* \\
& + e^{-ikr[\cos\theta-\theta_o)-1]-i\pi/4}(-ik)\cos(\theta-\theta_o)\sqrt{\frac{2}{\pi kr}}\mathcal{A}_o \Big\}
\end{aligned}$$

The first term in the integrand gives no contribution to the integral above because of periodicity. Since $\Im(if) = \Im(if^*)$, we get

$$\begin{aligned}
0 = & -\frac{2}{\pi}\int_0^{2\pi} |\mathcal{A}_o(\theta)|^2 d\theta \\
& + \Im \int_0^{2\pi} r d\theta \left\{ \mathcal{A}_o(-ik)\sqrt{\frac{2}{\pi kr}}[1+\cos(\theta-\theta_o)] e^{i\pi/4} e^{ikr(1-\cos(\theta-\theta_o))} \right\} \\
= & -\frac{2}{\pi}\int_0^{2\pi} |\mathcal{A}_o(\theta)|^2 d\theta \\
& - \Re \left\{ e^{-i\pi/4} \left[\mathcal{A}_o(k) r \sqrt{\frac{2}{\pi kr}} \int_0^{2\pi} d\theta [1+\cos(\theta-\theta_o)] e^{ikr(1-\cos(\theta-\theta_o))} \right] \right\}
\end{aligned}$$

For large kr the remaining integral can be found approximately by the method of stationary phase (see Appendix B), with the result

$$\int_0^{2\pi} d\theta [1+\cos(\theta-\theta_o)] e^{ikr(1-\cos(\theta-\theta_o))} \sim \sqrt{\frac{2\pi}{kr}} e^{i\pi/4} \tag{4.8}$$

We get finally

$$\int_0^{2\pi} |\mathcal{A}|^2 d\theta = -2\Re\mathcal{A}(\theta_o) \tag{4.9}$$

Thus the total scattered energy in all directions is related to the amplitude of the scattered wave in the forward direction. In atomic physics, where this theorem was originated (by Niels Bohr), measurement of the scattering amplitude in all directions is not easy. This theorem suggests an econmical alternative.

Homework For the same scatterer, consider two scattering problems ϕ_1 and ϕ_2. Show that

$$\mathcal{A}_1(\theta_2) = \mathcal{A}_2(\theta_1) \tag{4.10}$$

For general elastic waves, see Mei (1978) : Extensions of some identities in elastodynamics with rigid inclusions. *J . Acoust. Soc. Am.* 64(5), 1514-1522.

5 Diffraction by a thin barrier- parabolic approximation

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If the obstacle is large, there is always a shadow behind where the incident wave cannot penetrate deeply. The phenomenon of scattering by large obstacles is usually referred to as diffraction.

Diffraction of plane incident waves by a thin barrier is not only of interest to sound, but also to water waves scattered by a breakwater, and to elastic shear waves by a crack, ete. The exact solution by Sommerfeld is ammilestone in mathematicl physics. Here we shall give an approximate solution which reveals much of the physics. The method of approximation, due to V. Fock is of the boundary layer type called the *parabolic approximation*, and has been extended for modern applications in recent decades.

Referring to figure (5) let us make a crude division of the entire field. The illuminated zone I is dominated by the incident wave alone, the reflection zone II by the sum of the

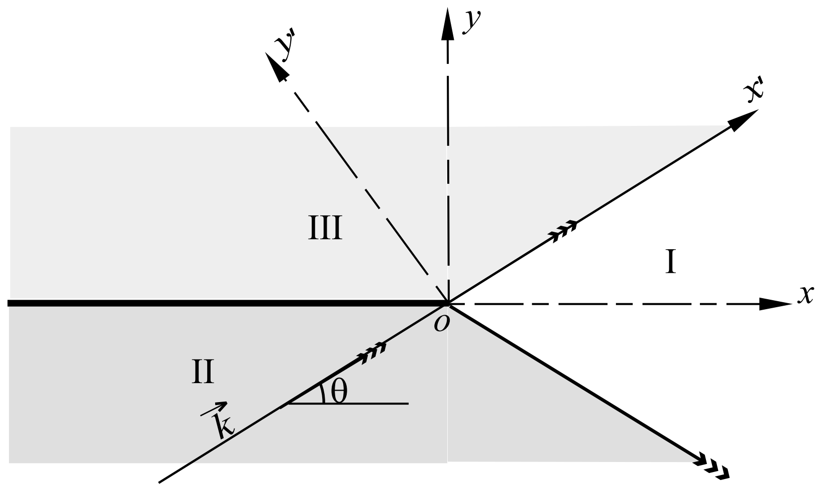

Figure 4: Wave zones near a thin barrier

incident and the reflected wave, and the shadow zone III. The boundaries of these zones are the rays touching the barrier tip. According to this crude picture of geometrical optics the solution is

$$\phi = \begin{cases} A_o \exp(ik\cos\theta x + ik\sin\theta y), & I; \\ A_o[\exp(ik\cos\theta x + ik\sin\theta y) + \exp(ik\cos\theta x - ik\sin\theta y)], & II \\ 0, & III \end{cases} \tag{5.1}$$

Clearly (5.1) is inadquate because the potential cannot be discontinuous across the boundaries. A remedy to ensure smooth transition is needed.

Consider the shadow boundary Ox'. Let us introduce a new cartesian coordinate system so that x' axis is along, while the y' axis is normal to, the shadow boundary. The relations between (x, y) and (x', y') are

$$x' = x\cos\theta + y\sin\theta, \quad y' = y\cos\theta - x\sin\theta \tag{5.2}$$

Thus the incident wave is simply

$$\phi_I = A_o e^{ikx'} \tag{5.3}$$

Following the chain rule of differentiation,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\phi}{\partial y'}\frac{\partial y'}{\partial x} = \cos\theta\frac{\partial\phi}{\partial x'} - \sin\theta\frac{\partial\phi}{\partial y'}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial\phi}{\partial y'}\frac{\partial y'}{\partial y} = \sin\theta\frac{\partial\phi}{\partial x'} + \cos\theta\frac{\partial\phi}{\partial y'}$$

we can show straightforwardly that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y'^2}$$

so that the Helmholtz equation is unchanged in form in the x', y' system.

We try to fit a boundary layer along the x' axis and expect the potential to be almost like a plane wave

$$\phi(x,', y') = A(x', y')e^{ikx'} \tag{5.4}$$

, but the amplitude is slowly modulated in both x' and y' directions. Substituting (5.4 into the Helmholtz equation, we get

$$e^{ikx'}\left\{\frac{\partial^2 A}{\partial x'^2}+2ik\frac{\partial A}{\partial x'}-k^2A+\frac{\partial^2 A}{\partial y'^2}+k^2A\right\}=0 \tag{5.5}$$

Expecting that the characteristic scale L_x of A along x' is much longer than a wavelength, $kL_x \gg 1$, we have

$$2ik\frac{\partial A}{\partial x'} \gg \frac{\partial^2 A}{\partial x'^2}$$

Hence we get as the first approximation the Schródinger equation

$$2ik\frac{\partial A}{\partial x'}+\frac{\partial^2 A}{\partial y'^2}\approx 0 \tag{5.6}$$

In this transition zone where the remaining terms are of comparable importance, hence the length scales must be related by

$$\frac{k}{x'} \sim \frac{1}{y'^2}, \quad \text{implying} \quad ky' \sim \sqrt{kx'}$$

Thus the transition zone is the interior of a parabola.

Equation (5.6) is of the parabolic type. The boundary conditions are

$$A(x,\infty)=0 \tag{5.7}$$

$$A(x,-\infty)=A_o \tag{5.8}$$

The initial condition is

$$A(0,y')=0, \quad \forall y' \tag{5.9}$$

Now the initial-boundary value for A has no intrinsic length scales except x', y' themselves. Therefore the condition $kL_x \gg 1$ means $kx' \gg 1$ i.e., far away from the tip. This problem is somwhat analogous to the problem of one-dimensional heat diffusion across a boundary. A convenient way of solution is the method of similarity.

Assume the solution

$$A = A_o f(\gamma) \tag{5.10}$$

where

$$\gamma = \frac{-ky'}{\sqrt{\pi k x'}} \tag{5.11}$$

is the similarity variable. We find upon subsitution that f satisfies the ordinary differential equation

$$f'' - i\pi\gamma f' = 0 \tag{5.12}$$

subject to the boundary conditions that

$$f \to 0, \quad \gamma \to -\infty; \qquad f \to 1, \quad \gamma \to \infty. \tag{5.13}$$

Rewriting (5.12) as

$$\frac{f''}{f'} = i\pi\gamma$$

we get

$$\log f' = i\pi\gamma/2 + \text{constant}.$$

One more integration gives

$$f = C \int_{-\infty}^{\gamma} \exp\left(\frac{i\pi u^2}{2}\right) du$$

Since

$$\int_0^{\infty} \exp\left(\frac{i\pi u^2}{2}\right) du = \frac{e^{i\pi/4}}{\sqrt{2}}$$

we get

$$C = \frac{e^{-i\pi/4}}{\sqrt{2}}$$

and

$$f = \frac{A}{A_o} = \frac{e^{-i\pi/4}}{\sqrt{2}} \int_{-\infty}^{\gamma} \exp\left(\frac{i\pi u^2}{2}\right) du = \frac{e^{-i\pi/4}}{\sqrt{2}} \left\{ \frac{e^{i\pi/4}}{\sqrt{2}} + \int_0^{\gamma} \exp\left(\frac{i\pi u^2}{2}\right) du \right\} \tag{5.14}$$

Defining the cosine and sine Fresnel integrals by

$$C(\gamma) = \int_0^{\gamma} \cos\left(\frac{\pi v^2}{2}\right) dv, \quad S(\gamma) = \int_0^{\gamma} \sin\left(\frac{\pi v^2}{2}\right) dv \tag{5.15}$$

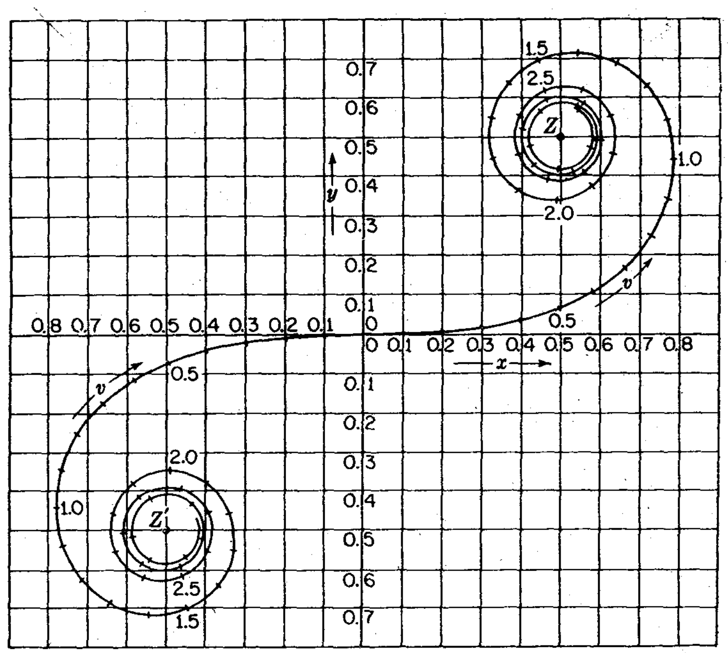

Fig. 18*N*. Cornu's spiral, a plot of the Fresnel integrals.

Figure 5: Ciornu's spiral as a function of γ.

we can then write

$$\frac{e^{-i\pi/4}}{\sqrt{2}}\left\{\left[\frac{1}{2}+C(\gamma)\right]+i\left[\frac{1}{2}+S(\gamma)\right]\right\} \tag{5.16}$$

In the complex plane the plot of $C(\gamma)+iS(\gamma)$ vs. γ is the famous Cornu's spiral, shown in figure (5).

The wave intensity is given by

$$\frac{|A|^2}{A_o^2}=\frac{1}{2}\left\{\left[\frac{1}{2}+C(\gamma)\right]^2+\left[\frac{1}{2}+S(\gamma)\right]^2\right\} \tag{5.17}$$

Since $C, S \to 0$ as $\gamma \to -\infty$, the wave intensity diminshes to zero gradually into the shadow. However, $C, S \to 1/2$ as $\gamma \to \infty$ in an oscillatory manner. Hence the wave intensity oscillates while approaching to unity asymptotically, as shown in figure 5. In optics these oscillations show up as alternately light and dark diffraction bands.

In more complex propagation problems, the parabolic approximation can simplify the numerical task in that an elliptic boundary value problem involving an infinite domain

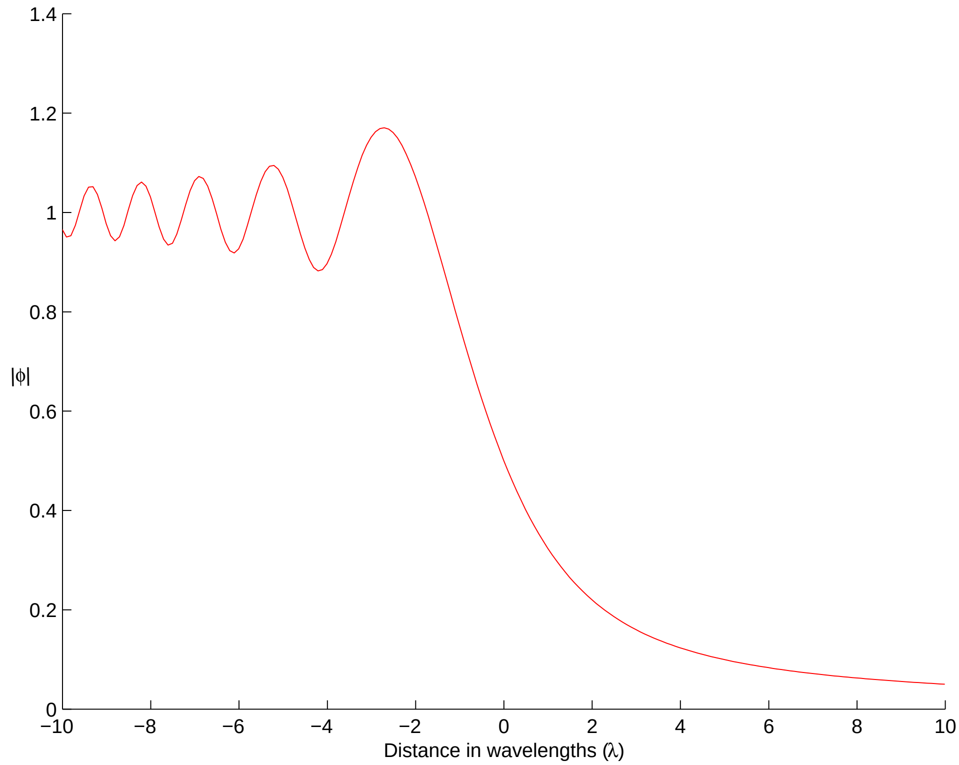

Figure 6: Wave intensity variation near the shadow boundary

is reduced to an initial boundary value problem. One can use Crank-Nicholson scheme to march in "time", i.e., x'.

Homework Find by the parabolic approximation the transition solution along the edge of the reflection zone.

6 Diffraction by a Wedge – An Exact Theory

Refs. Stoker:

In the preceding section we gave an approximate theory by parabolic approximation. Extending the theory of Sommerfeld, Peters and Stoker (1954) have given an exact theory for the general case of a wedge, see Stoker (1957). In the original work a series solution was first obtained by finite Fourier transform. The resulting series was then summed in terms of integrals from which approximate informtation was then extracted by some intricate asymptotic analysis. With the power of the modern computer, it is

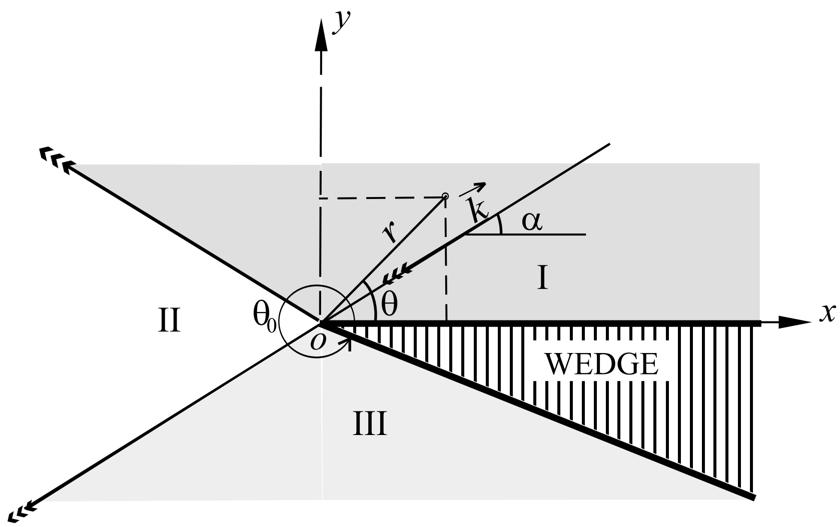

Figure 7: Waves scattering by a wedge

more straightforward to get quantitative results from direct numerical calculation of the series, as exemplified by Chen (1982). To facilitate the understanding of the physics, these results are presented in animated form in "link-simulations". The exact theory will then be compared with the parabolic approximation.

Refering to Figure 6 the entire fluid region can be divided into three zones according to the crude picture of geometrical optics. I: the zone of incident and reflected waves, II : the zone of incident waves and III ; the shadow. To ensure smooth transition in all zones there is also the diffracted (or scattered) waves. The total potential can be expressed in a compact form by

$$\phi = \phi_o(r,\theta) + \phi_s(r,\theta), \quad \text{for all } \ 0 < \theta < \nu\pi \tag{6.18}$$

where ϕ_o is defined by

$$\phi_o(r,\theta) = \begin{cases} \phi_i + \phi_r & \pi - \alpha > \theta > 0, \quad \text{in I;} \\ \phi_i & \pi + \alpha > \theta > \pi - \alpha, \quad \text{in II;} \\ 0 & \theta_0 > \theta > \pi + \alpha, \quad \text{in III.} \end{cases} \tag{6.19}$$

and ϕ_s is the scattered (or diffracted) waves. Both the incident wave

$$\phi_i = e^{-ikr\cos(\theta-\alpha)} \tag{6.20}$$

and the reflected wave

$$\phi_r = e^{-ikr\cos(\theta+\alpha)} \tag{6.21}$$

are known, where α denotes the angle of incidence. The unknown scattered wave ϕ_s must satisfy the radiation condition and behaves as an outgoing wave at infinity, i.e.,

$$\lim_{r\to 0}\sqrt{r}\left(\frac{\partial\phi_s}{\partial r}-ik\phi_s\right)=0 \tag{6.22}$$

or

$$\phi_s\sim\frac{\mathcal{A}(\theta)e^{ikr}}{\sqrt{kr}}\qquad at\;\; r\to\infty \tag{6.23}$$

6.1 Solution by finite Fourier Transform

Let us introduce the finite cosine transform of ϕ defined by

$$\bar{\phi}(kr,n)=\int_0^{\nu\pi}\phi(kr,\theta)\cos\frac{n\theta}{\nu}d\theta \tag{6.24}$$

where n=0, 1, 2, ... are integers. the inverse transform is

$$\phi(r,\theta)=\frac{1}{\nu\pi}\bar{\phi}(r,0)+\frac{2}{\nu\pi}\sum_{n=1}^{\infty}\bar{\phi}(r,n)\cos\frac{n\theta}{\nu} \tag{6.25}$$

It is easily recognized that the transform is equivalent to expansion in cosine series. Applying the finite cosine transform and using the boundary conditions on the walls,

$$\begin{aligned}\overline{\frac{\partial^2\phi}{\partial\theta^2}} &= \int_0^{\nu\pi}\frac{\partial^2\phi}{\partial\theta^2}\cos\frac{n\theta}{\nu}d\theta\\ &= \left[\frac{\partial\phi}{\partial\theta}\cos\frac{n\theta}{\nu}\right]_{\theta=0}^{\theta=\nu\pi}+\frac{n}{\nu}\int_0^{\nu\pi}\frac{\partial\phi}{\partial\theta}\sin\frac{n\theta}{\nu}d\theta\\ &= \left[\frac{n}{\nu}\phi\sin\frac{n\theta}{\nu}\right]_{\theta=0}^{\theta=\nu\pi}-\frac{n^2}{\nu^2}\int_0^{\nu\pi}\phi\cos\frac{n\theta}{\nu}d\theta\\ &= -\frac{n^2}{\nu^2}\int_0^{\nu\pi}\phi\cos\frac{n\theta}{\nu}d\theta\end{aligned} \tag{6.26}$$

Eq. (1.21) becomes

$$r^2\frac{\partial^2\bar{\phi}}{\partial r^2}+r\frac{\partial\bar{\phi}}{\partial r}+\left[(kr)^2-\left(\frac{n}{\nu}\right)^2\right]\bar{\phi}=0 \tag{6.27}$$

The general solution finite at the origin is

$$\bar{\phi}(kr,n)=a_nJ_{n/\nu}(kr) \tag{6.28}$$

where the coefficient's $a_n, n=0,1,2,3,...$ are to be determined.

The finite cosine transform of (6.18) reads

$$a_n J_{n/\nu}(kr) = \int_0^{\nu\pi} \phi_s \cos\frac{n\theta}{\nu} d\theta + \int_0^{\nu\pi} \phi_o \cos\frac{n\theta}{\nu} d\theta \tag{6.29}$$

or

$$\bar{\phi}_s = a_n J_{n/\nu}(kr) - \bar{\phi}_0 \tag{6.30}$$

Applying the operator $\lim_{r\to\infty}\sqrt{r}(\partial/\partial r - ik)$ to both sides of (6.29), and using the Sommerfeld radiation condition (6.22), we have

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\left[a_n J_{n/\nu}(kr) - \int_0^{\nu\pi} \phi_0 \cos\frac{n\theta}{\nu} d\theta\right] = 0 \tag{6.31}$$

We now perform some asymptotic analysis to evaluate a_n.

First, for large kr we have

$$J_{n/\nu}(kr) \sim \sqrt{\frac{2}{\pi kr}}\cos\left(kr - \frac{n\pi}{2\nu} - \frac{\pi}{4}\right) \tag{6.32}$$

It follows that

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) J_{n/\nu}(kr) = \sqrt{\frac{2k}{\pi}} e^{-i\left(kr - \frac{n\pi}{2\nu} + \frac{\pi}{4}\right)} \tag{6.33}$$

Second, we substitute ϕ_o from (6.20) and (6.21) to rewrite the integral of ϕ_o as

$$\begin{aligned}\int_0^{\nu\pi} \phi_0 \cos\frac{n\theta}{\nu} d\theta \quad &= \overbrace{\int_0^{\pi-\alpha} e^{-ikr\cos(\theta-\alpha)}\cos\frac{n\theta}{\nu} d\theta}^{1} + \overbrace{\int_0^{\pi-\alpha} e^{-ikr\cos(\theta+\alpha)}\cos\frac{n\theta}{\nu} d\theta}^{2} \\ &+ \overbrace{\int_{\pi-\alpha}^{\pi+\alpha} e^{-ikr\cos(\theta-\alpha)}\cos\frac{n\theta}{\nu} d\theta}^{3} \end{aligned} \tag{6.34}$$

Each of the integrals above scan be evaluated for large kr by the method of stationary phase. The details are given in the appendix C; only the results are cited below.

The first integral is approximately

$$I_1(\theta) = \cos\left(\frac{n\alpha}{\nu}\right) e^{-ikr + \frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} + O\left(\frac{1}{kr}\right) \tag{6.35}$$

from which

$$\begin{aligned}&\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\int_0^{\pi-\alpha} e^{-ikr\cos(\theta-\alpha)}\cos\frac{n\theta}{\nu} d\theta \\ &\quad = \lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r} - ik\right)\left\{\cos\left(\frac{n\alpha}{\nu}\right) e^{-ikr+\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}}\right\} \\ &\quad = 2\sqrt{2\pi k}\cos\left(\frac{n\alpha}{\nu}\right) e^{-ikr - \frac{i\pi}{4}} \end{aligned} \tag{6.36}$$

where we have used $i = e^{i\pi/2}$. By similar analysis the second integral is found to be

$$I_2(\theta) \approx \frac{1}{2} \cos\left(\frac{n(\pi - \alpha)}{\nu}\right) e^{ikr - \frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{6.37}$$

It follows that

$$\begin{aligned}
&\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \int_0^{\pi-\alpha} e^{-ikr\cos(\theta+\alpha)} \cos\frac{n\theta}{\nu} d\theta \\
&= \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \left\{\frac{1}{2}\cos(\frac{n(\pi-\alpha)}{\nu}) e^{ikr - \frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}}\right\} \\
&= 0
\end{aligned} \tag{6.38}$$

Finally the third integral is approximately

$$I_3(\theta) \approx \frac{1}{2} \cos(\frac{n(\pi + \alpha)}{\nu}) e^{ikr - \frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{6.39}$$

hence

$$\begin{aligned}
&\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \int_0^{\pi+\alpha} e^{-ikr\cos(\theta+\alpha)} \cos\frac{n\theta}{\nu} d\theta \\
&= \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \left\{\frac{1}{2}\cos(\frac{n(\pi+\alpha)}{\nu}) e^{ikr - \frac{i\pi}{4}} \left[\frac{2\pi}{kr}\right]^{\frac{1}{2}}\right\} \\
&= 0
\end{aligned} \tag{6.40}$$

In summary, only the first integral associated with the incident wave furnishes a nonvanishing contribution to the expansion coefficients, i.e.,

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial}{\partial r} - ik\right) \int_0^{\nu\pi} \phi_0 \cos\frac{n\theta}{\nu} d\theta \sim 2\sqrt{2\pi k} \cos\frac{n\alpha}{\nu} e^{-i\left(kr + \frac{\pi}{4}\right)} \tag{6.41}$$

With this result we get by substituting (6.33) and (6.41) into (6.31), the coefficients a_n are found

$$a_n = 2\pi \cos\frac{n\alpha}{\nu} e^{-i\frac{n\pi}{2\nu}} \tag{6.42}$$

By inverse transform, (6.25), we get the exact solution,

$$\phi(r,\theta) = \frac{2}{\nu}\left[J_0(kr) + 2\sum_{n=1}^{\infty} e^{-i\frac{n\pi}{2\nu}} J_{n/\nu}(kr) \cos\frac{n\alpha}{\nu} \cos\frac{n\theta}{\nu}\right] \tag{6.43}$$

6.2 Two limiting cases

(1) A thin barrier. Let the wedge angle be 0 by setting $\nu = 2$. Equation (6.43) then becomes

$$\phi(r,\theta) = J_0(kr) + 2\sum_{n=1}^{\infty} e^{-i\frac{n\pi}{4}} J_{n/2}(kr)\cos\frac{n\alpha}{2}\cos\frac{n\theta}{2} \tag{6.44}$$

(see Stoker (1957)).

(2) An infinite wall extending from $x = -\infty$ to ∞. Water occupying only the half plane of $y \geq 0$ and the wedge angle is 180 degrees. The diffracted wave is absent from the solution, and the total wave is only the sum of the incident and reflected waves :

$$\phi(r,\theta) = e^{-ikr\cos(\theta-\alpha)} + e^{-ikr\cos(\theta+\alpha)} \tag{6.45}$$

By employing the partial-wave expansion theorm, (Abramowitz and Stegun 1964), the preceding equaion becomes

$$\phi(r,\theta) = 2\left[J_0(kr) + 2\sum_{n=1}^{\infty}(-i)^n J_n(kr)\cos n\alpha\cos n\theta\right] \tag{6.46}$$

which agree with (6.43) for $\nu = 1$.

For sample results, click website.

6.3 Comparison with Parabolic Approximation

to be written

A Partial wave expansion

A useful result in wave theory is the expansion of the plane wave in a Fourier series of the polar angle θ. In polar coordinates the spatial factor of a plane wave of unit amplitude is

$$e^{ikx} = e^{ikr\cos\theta}.$$

Consider the following product of exponential functions

$$e^{zt/2}e^{-z/2t} = \left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{zt}{2}\right)^n\right]\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-z}{2t}\right)^n\right]$$

$$\sum_{-\infty}^{\infty} t^n\left[\frac{(z/2)^n}{n!} - \frac{(z/2)^{n+2}}{1!(n+1)!} + \frac{(z/2)^{n+4}}{2!(n+2)!} + \cdots + (-1)^r\frac{(z/2)^{n+2r}}{r!(n+r)!} + \cdots\right].$$

The coefficient of t^n is nothing but $J_n(z)$, hence

$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{-\infty}^{\infty} t^n J_n(z).$$

Now we set

$$t = ie^{i\theta} \quad z = kr.$$

The plane wave then becomes

$$e^{ikx} = \sum_{N=-\infty}^{\infty} e^{in(\theta+\pi/2} J_n(z).$$

Using the fact that $J_{-n} = (-1)^n J_n$, we finally get

$$e^{ikx} = e^{ikr\cos\theta} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kr) \cos n\theta, \tag{A.1}$$

where ϵ_n is the Jacobi symbol. The above result may be viewed as the Fourier expansion of the plane wave with Bessel functions being the expansion coefficients. In wave propagation theories, each term in the series represents a distinct angular variation and is called a *partial wave*.

Using the orthogonality of $\cos n\theta$, we may evaluate the Fourier coefficient

$$J_n(kr) = \frac{2}{\epsilon_n i^n \pi} \int_0^{\pi} e^{ikr\cos\theta} \cos n\theta d\theta, \tag{A.2}$$

which is one of a host of integral representations of Bessel functions.

B Approximation of an integral

Consider the integral

$$\int_0^{2\pi} d\theta [1 + \cos(\theta - \theta_o)] e^{ikr(1-\cos(\theta-\theta_o))}$$

For large kr the stationary phase points are found from

$$\frac{\partial}{\partial\theta}[1 - \cos(\theta - \theta_o)] = \sin(\theta - \theta_o) = 0$$

or $\theta = \theta_o, \theta_o + \pi$ within the range $[0, 2\pi]$. Near the first stationary point the integrand is dominated by

$$2\mathcal{A}(\theta_o) e^{ikt(\theta-\theta_o)^2/2}.$$

When the limits are approximated by $(-\infty,\infty)$, the inegral can be evaluated to give

$$\mathcal{A}(\theta_o)\int_{-\infty}^{\infty} e^{ikr\theta^2/2}d\theta = \sqrt{\frac{2\pi}{kr}}e^{i\pi/4}\mathcal{A}(\theta_o)$$

Near the second stationary point the integral vanishes since $1+\cos(\theta-\theta_o) == 1-1 = 0$. Hence the result (4.8) follows.

C Asymptotic evaluation of integrals

For the first integral I_1, we take the phase to be $f_1(\theta) = k\cos(\theta-\alpha)$. The points of stationary phase must be found from

$$f_1'(\theta) = -k\sin(\theta-\alpha) = 0, \tag{C.1}$$

hence $\theta = \alpha, \alpha\pm\pi$. Only the first at $\theta_1 = \alpha$ lies in the range of integration $(0,\pi-\alpha)$ and is the stationary point. Since

$$f_1''(\theta_1) = -k\cos(\theta_1-\alpha) = -k < 0 \tag{C.2}$$

the integral is approximately

$$I_1(\theta) \approx \cos(\frac{n\theta_1}{\nu})e^{-ikr\cos(\theta_1-\alpha)+\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} = \cos(\frac{n\alpha}{\nu})e^{-ikr+\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{C.3}$$

For the second integral I_2, we take the phase to be $f_2(\theta) = k\cos(\theta+\alpha)$. The stationary phase point must be the root of

$$f_2'(\theta) = -k\sin(\theta+\alpha) = 0 \tag{C.4}$$

or $\theta = -\alpha, -\alpha\pm\imath$. The stationary point is at $\theta_2 = \pi-\alpha$ which is the the upper limit of integration. Since

$$f_2''(\theta_2) = -k\cos(\theta_2+\alpha) = k > 0 \tag{C.5}$$

I_2 is approximately

$$I_2(\theta) \approx \frac{1}{2}\cos(\frac{n\theta_2}{\nu})e^{-ikr\cos(\theta_2+\alpha)-\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} = \frac{1}{2}\cos(\frac{n(\pi-\alpha)}{\nu})e^{ikr-\frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{C.6}$$

Lastly for the third integral I_3, the phase is $f_3(\theta) = k\cos(\theta - \alpha)$. The point of stationary phase is found from

$$f_3'(\theta) = -k\sin(\theta - \alpha) = 0 \tag{C.7}$$

or $\theta = \pi, \pm\pi + \alpha$. Only the point $\theta_3 = \pi + \alpha$ is acceptable and coincides with the upper limit of integration. Since

$$f_3''(\theta_3) = -k\cos(\theta_3 - \alpha) = k > 0, \tag{C.8}$$

I_3 is approximately

$$I_3(\theta) \approx \frac{1}{2}\cos(\frac{n(\pi + \alpha)}{\nu})e^{ikr - \frac{i\pi}{4}}\left[\frac{2\pi}{kr}\right]^{\frac{1}{2}} \tag{C.9}$$