I-campus project

School-wide Program on Fluid Mechanics

Modules on Waves in fluids

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CHAPTER FOUR

WAVES DUE TO MOVING DISTURBANCES

2 D SHIP WAVES

RADIATION DUE TO OSCILLATING PRESSURE

SHIP WAVES

[Ref]: Lecture notes on *Surface Wave Hydrodynamics* by Theodore T.Y. WU, Calif. Inst.Tech.

Lighhill, Waves in Fluids

1 Wave resistance of a two-dimensional obstacle

As an application of the information gathered so far, let us examine the wave resistance on a two dimensional body steadily advancing parallel to the free surface. Let the body speed be U from right to left and the sea depth be constant.

Due to two-dimensionality, waves generated must have crests parallel to the axis of the body (y axis). After the steady state is reached, waves that keep up with the ship must have the phase velocity equal to the body speed. In the coordinate system fixed on the body, the waves are stationary. Consider first capillary -gravity waves in deep water $\lambda_* = \lambda/\lambda_m = O(1)$ and $kh \gg 1$. Equating $U = c$ we get from the normalized dispersion relation

$$U_*^2 = c_*^2 = \frac{1}{2}\left(\lambda_* + \frac{1}{\lambda_*}\right) \tag{1.1}$$

where $U_* \equiv U/c_m$. Hence

$$\lambda_*^2 - 2c_*^2\lambda_* + 1 = 0 = (\lambda_* - \lambda_{*1})(\lambda_* - \lambda_{*2})$$

which can be solved to give

$$\begin{bmatrix} \lambda_{*1} \\ \lambda_{*2} \end{bmatrix} = c_*^2 \pm \left(c_*^4 - 1\right)^{1/2} \tag{1.2}$$

and

$$\lambda_{*1} = \frac{1}{\lambda_{*2}} \tag{1.3}$$

Thus, as long as $c_* = U_* > 1$ two wave trains are present: the longer gravity wave with length λ_{*1}, and the shorter capillary wave with length λ_{*2}. Since $c_{g_1} < c = U$ and $c_{g_2} > c = U$, and energy must be sent from the body, the longer gravity waves must follow, while the shorter capillary waves stay ahead of, the body.

Balancing the power supply by the body and the power flux in both wave trains, we get

$$Rc = (c - c_{g_1})\bar{E}_1 + (c_{g_2} - c)\bar{E}_2 \tag{1.4}$$

Recalling that

$$\frac{c_g}{c} = \frac{1}{2}\frac{\lambda_*^2 + 3}{\lambda_*^2 + 1}$$

we find,

$$1 - \frac{c_g}{c} = 1 - \frac{1}{2}\left(1 + \frac{2}{\lambda_*^2 + 1}\right) = \frac{1}{2} - \frac{1/\lambda_*}{\lambda_* + 1/\lambda_*} = \frac{1}{2} - \frac{1/\lambda_*}{2c^2}$$

For the longer wave we replace c_g by $c_{g_{*1}} c_m$ and λ_{*1} in the preceding equation, and use (1.2), yielding

$$1 - \frac{c_{g*1}}{c_*} = \left(1 - c_*^4\right)^{1/2} \tag{1.5}$$

Similarly we can show that

$$\frac{c_{g*2}}{c_*} - 1 = \left(1 - c_*^4\right)^{1/2} = 1 - \frac{c_{g*1}}{c_*} \tag{1.6}$$

Since

$$\bar{E}_1 = \frac{\rho g A_1^2}{2}\left(1 + \frac{1}{\lambda_{*1}^2}\right) = \frac{\rho g A_1^2}{2}\frac{1}{\lambda_{*1}}\left(\lambda_{*1} + \frac{1}{\lambda_{*1}}\right) = \rho g A_1^2 \lambda_{*2} c_*^2, \tag{1.7}$$

we get finally

$$R = \frac{1}{2}\rho g \left(\lambda_{*2} A_1^2 + \lambda_{*1} A_2^2\right)(c_*^2 - 1)^{1/2} = \frac{1}{2}\rho g \left(\lambda_{*2} A_1^2 + \lambda_{*1} A_2^2\right)(U_*^2 - 1)^{1/2} \tag{1.8}$$

Note that when $U_* = 1$, the two waves become the same; no power input from the body is needed to maintain the single infinite train of waves; the wave resistance vanishes. When $U_* < 1$, no waves are generated; thedisturbance is purely local and there is also no wave resistance. To get the magnitude of R one must solve the boundary value problem for the wave amplitudes A_1, A_2 which are affected by the size (relative to the wavelengths), shape and depth of submergence.

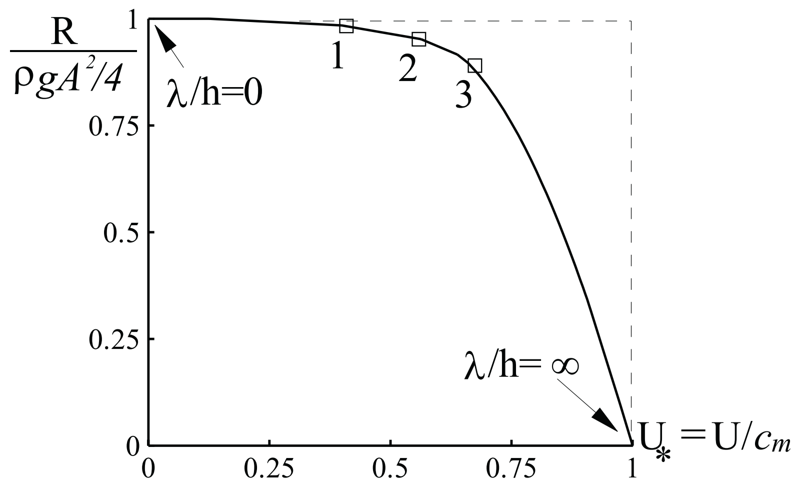

Figure 1: Dependence of wave resistance on speed for pure gravity waves

When the speed is sufficiently high, pure gravity waves are generated behind the body. Power balance then requires that

$$R = \left(1 - \frac{c_g}{U}\right)\bar{E} = \frac{\rho g A^2}{2}\left(\frac{1}{2} - \frac{2kh}{\sinh 2kh}\right) \tag{1.9}$$

The wavelength generated by the moving body is given implicitly by

$$\frac{U}{\sqrt{gh}} = \left(\frac{\tanh kh}{kh}\right)^{1/2} \tag{1.10}$$

When $U \approx \sqrt{gh}$ the waves generatesd are very long, $kh \ll 1$, $c_g \to c = \sqrt{gh}$, and the wave resistance drops to zero. When $U \ll \sqrt{gh}$, the waves are very short, $kh \gg 1$,

$$R \approx \frac{\rho g A^2}{4} \tag{1.11}$$

For intermediate sppeds the dependence of wave resistance on speed is plotted in figure (1).

2 Radiation of surface waves forced by an oscillating pressure

We demonstrate the reasoning which is typical in many similar radiation problems.

The governing equations are

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0, \quad -\infty < z < 0. \tag{2.1}$$

with the kinematic boundary condition

$$\phi_z = \zeta_t, \quad z = 0 \tag{2.2}$$

and the dynamic boundary condiition

$$\frac{p_a}{\rho} + \phi_t + g\zeta = 0 \tag{2.3}$$

where p_a is the prescribed air pressure. Eliminating the free surface displacement we get

$$\phi_{tt} + g\phi_z = -\frac{(p_a)_t}{\rho}, \quad z = 0. \tag{2.4}$$

Let us consider only sinusoidal time dependence:

$$p_a = P(x)e^{-i\omega t} \tag{2.5}$$

and assume

$$\phi(x, z, t) = \Phi(x, z)e^{-i\omega t}, \quad \zeta(x, t) = \eta(x)e^{-i\omega t} \tag{2.6}$$

then the governing equations become

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{yy} = 0, \quad -\infty < z < 0. \tag{2.7}$$

$$\Phi_z = -i\omega\eta, \quad z = 0 \tag{2.8}$$

and

$$\Phi_z - \frac{\omega^2}{g}\Phi = \frac{i\omega}{\rho g}P(x), \quad z = 0. \tag{2.9}$$

Define the Fourier tranoform and its inverse by

$$\bar{f}(\alpha) = \int_{-\infty}^{\infty} dx\, f(x), \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, \bar{f}(\alpha), \tag{2.10}$$

We then get the transforms of (2.1) and (2.4)

$$\bar{\Phi}_{zz} - \alpha^2\bar{\Phi} = 0, \quad z < 0 \tag{2.11}$$

subject to

$$\bar{\Phi}_z - \frac{\omega^2}{g}\bar{\Phi} = \frac{i\omega}{\rho g}\bar{P}, \quad z = 0. \tag{2.12}$$

The solution finite at $z \sim -\infty$ for all α is

$$\bar{\Phi} = Ae^{|\alpha|z}$$

To satisfy the free surface condition

$$|\alpha|A - \frac{\omega^2}{g}A = \frac{i\omega\bar{P}}{\rho g}$$

hence

$$A = \frac{\frac{i\omega\bar{P}}{\rho g}}{|\alpha| - \omega^2/g}$$

or

$$\begin{aligned}
\Phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x} e^{|\alpha|z} \frac{\frac{i\omega\bar{P}}{\rho g}}{|\alpha| - \omega^2/g} \\
&= \frac{i\omega}{\rho g}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha e^{i\alpha x} e^{|\alpha|z}\int_{-\infty}^{\infty} dx'\, e^{-i\alpha x'} P(x') \frac{1}{|\alpha| - \omega^2/g}, \\
&= \frac{i\omega}{\rho g}\int_{-\infty}^{\infty} dx'\, P(x') \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha(x-x')} e^{|\alpha|z} \frac{1}{|\alpha| - \omega^2/g}
\end{aligned} \tag{2.13}$$

Let

$$k = \frac{\omega^2}{g} \tag{2.14}$$

we can rewrite (2.13) as

$$\Phi = \frac{i\omega}{\rho g}\int_{-\infty}^{\infty} dx'\, P(x') \frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha(x - x'))}{\alpha - k} \tag{2.15}$$

The final formal solution is

$$\phi = \frac{i\omega}{\rho g} e^{-i\omega t}\int_{-\infty}^{\infty} dx'\, P(x') \frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha(x - x'))}{\alpha - k} \tag{2.16}$$

If we chose

$$P(x') = P_o\delta(x') \tag{2.17}$$

then

$$\Phi \to \mathcal{G}(x, z) = \frac{i\omega P_o}{\rho g}\frac{1}{\pi}\int_0^{\infty} d\alpha\, e^{\alpha z} \frac{\cos(\alpha x)}{\alpha - k} \tag{2.18}$$

is clearly the response to a concentrated surface pressure and the response to a pressure distribution (2.16) can be written as a superposition of concentrated loads over the free surface,

$$\phi = \int_{-\infty}^{\infty} dx' \, P(x') G(x - x', z). \tag{2.19}$$

where

$$G(x, z, t) = \frac{i\omega P_o}{\rho g} e^{-i\omega t} \frac{1}{\pi} \int_0^\infty d\alpha \, e^{\alpha z} \frac{\cos(\alpha x)}{\alpha - k} \tag{2.20}$$

In these results, e.g., (2.20), the Fourier integral is so far undefined since the integrand has a real pole at $\alpha = k$ which is on the path of integration. To make it mathematically defined we can chose the principal value, deform the contour from below or from above the pole as shown in figure (2). This indefiniteness is due to the assumption of quasi

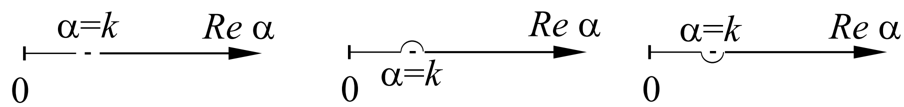

Figure 2: Possible paths of integration

steady state where the influence of the initial condition is no longer traceable. We must now impose the radiation condition that waves must be outgoing as $x \to \infty$. This condition can only be satisfied if we deform the contour from below. Denoting this contour by Γ, we now manipulate the integral to exhibit the behavior at infinity, and to verify the choice of path. For simplicity we focus attention on $\mathcal{G}$. Due to symmetry, it suffices to consider $x > 0$. Rewriting,

$$\begin{aligned} G(x, z, t) &= \frac{i\omega P_o}{\rho g} e^{-i\omega t} \frac{1}{2\pi} (I_1 + I_2) \\ &= \frac{i\omega P_o}{\rho g} e^{-i\omega t} \frac{1}{2\pi} \int_\Gamma d\alpha \, e^{\alpha z} \left[\frac{e^{i\alpha x}}{\alpha - k} + \frac{e^{-i\alpha x}}{\alpha - k} \right] \end{aligned} \tag{2.21}$$

Consider the first integral in (2.21). In order that the first integral converges for large $|\alpha|$, we close the contour by a largfe circular arc in the upper half plane, as shown in figure (3), where $\Im\alpha > 0$ along the arc. The term

$$e^{i\alpha x} = e^{i\Re\alpha x} e^{-\Im\alpha x}$$

is exponentially small for positive x. Similarly, for the second integral we must chose the contour by a large circular arc in the lower half plane as shown in figure (4).

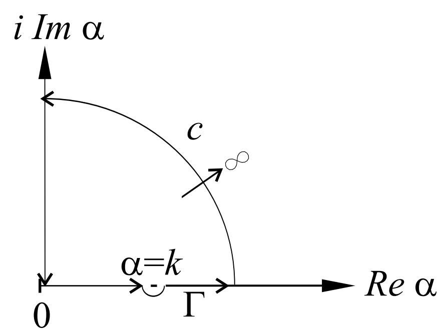

Figure 3: Closed contour in the upper half plane

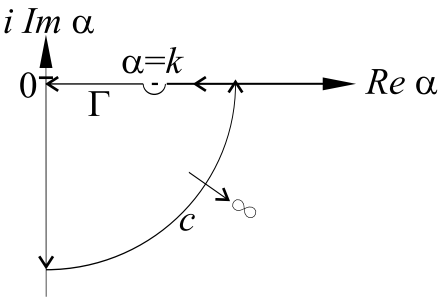

Figure 4: Closed contour in the lower half plane

Back to the first integral in (2.21)

$$I_1 = \int_\Gamma d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} \tag{2.22}$$

The contour integral is

$$\begin{aligned} \oint d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} &= \int_\Gamma d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} + \int_C d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} + \int_{i\infty}^0 d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} \\ &= I_1 + 0 + \int_{i\infty}^0 d\alpha \frac{e^{i\alpha x} e^{\alpha z}}{\alpha - k} \end{aligned}$$

The contribution by the circular arc C vanishes by Jordan's lemma. The left hand side is

$$LHS = 2\pi i e^{ikx} e^{kz} \tag{2.23}$$

by Cauchy's residue theorem. By the change of variable $\alpha = i\beta$, the right hand side becomes

$$RHS = I_1 + i \int_\infty^0 d\beta \frac{e^{-\beta x} e^{i\beta z}}{i\beta - k}$$

Hence

$$I_1 = 2\pi i e^{ikx} e^{kz} + i \int_0^\infty d\beta \frac{e^{-\beta x} e^{i\beta z}}{i\beta - k} \tag{2.24}$$

Now consider I_2

$$I_2 = \int_\Gamma d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k} \tag{2.25}$$

and the contour integral along the contour closed in the lower half plane,

$$-\oint d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k} = I_2 + 0 + \int_0^\infty d\alpha \frac{e^{-i\alpha x} e^{\alpha z}}{\alpha - k}$$

Again no contribution comes from the circular arc C. Now the pole is outside the contour hence $LHS = 0$. Let $\alpha = -i\beta$ in the last integral we get

$$I_2 = -i \int_0^\infty d\beta \frac{e^{-\beta x} e^{-i\beta y}}{-i\beta - k} \tag{2.26}$$

Adding the results (2.24) and (2.26).,

$$\begin{aligned} I_1 + I_2 &= 2\pi i e^{ikx} e^{kz} + \int_0^\infty d\beta \left(\frac{ie^{-\beta x} e^{i\beta z}}{i\beta - k} - \frac{ie^{-\beta x} e^{-i\beta z}}{-i\beta - k} \right) \\ &= 2\pi i e^{ikx} e^{kz} + 2 \int_0^\infty d\beta \frac{e^{-\beta x}}{\beta^2 + k^2} (\beta \cos \beta y + k \sin \beta y) \end{aligned} \tag{2.27}$$

Finally, the total potential is

$$\begin{aligned} G(x, z, t) &= -\frac{\omega}{\rho g} e^{-i\omega t} \left(\frac{1}{2\pi i} (I_1 + I_2) \right) e^{-i\omega t} \\ &= -\frac{\omega}{\rho g} e^{-i\omega t} \left\{ e^{ikx} e^{kz} + \frac{1}{\pi} \int_0^\infty d\beta \frac{e^{-\beta x}}{\beta^2 + k^2} (\beta \cos \beta y + k \sin \beta y) \right\} \end{aligned} \tag{2.28}$$

The first term gives an outgoing waves. For a concentrated load with amplitude P_o, the displacement amplitude is $P_o/\rho g$. The integral above represent local effects important only near the applied pressure. If the concentrated load is at $x = x'$, one simply replaces x by $x - x'$ everywhere.

3 The Kelvin pattern of ship waves

Anyone flying over a moving ship must be intrigued by the beautiful pattern in the ship's wake. The theory behind it was first completed by Lord Kelvin, who invented the method of stationary phase for the task. Here we shall give a physical/geometrical derivation of the key results (lecture notes by T. Y. Wu, Caltech).

Consider first two coordinate systems. The first $\mathbf{r} = (x, y, z)$ moves with ship at the uniform horizontal velocity $\mathbf{U}$. The second $\mathbf{r}' = (x', y', z)$ is fixed on earth so that water is stationary while the ship passes by at the velocity $\mathbf{U}$. The two systems are related by the Galilean transformation,

$$\mathbf{r}' = \mathbf{r} + \mathbf{U}t \tag{3.1}$$

A train of simple harmonic progressive wave

$$\zeta = \Re \{A \exp[i(\mathbf{k} \cdot \mathbf{r}' - \omega t)]\} \tag{3.2}$$

in the moving coordinates should be expressed as

$$\begin{aligned} \zeta &= \Re \{A \exp[i\mathbf{k} \cdot (\mathbf{r} - \mathbf{U}t) - i\omega t]\} = \Re \{A \exp[i\mathbf{k} \cdot \mathbf{r} - i(\omega - \mathbf{k}.\mathbf{U})t]\} \\ &= \Re \{A \exp[i\mathbf{k} \cdot \mathbf{r} - i\sigma t]\} \end{aligned} \tag{3.3}$$

in the stationary coordinates. Therefore the apparent frequency in the moving coordinates is

$$\sigma = \omega - \mathbf{k} \cdot \mathbf{U} \tag{3.4}$$

The last result is essentially the famous Doppler's effect. To a stationary observer, the whistle from an approaching train has an increasingly high pitch, while that from a leaving train has a decreasing pitch.

If a ship moves in very deep water at the constant speed $-\mathbf{U}$ in stationary water, then relative to the ship, water appears to be washed downstream at the velocity $\mathbf{U}$. A stationary wave pattern is formed in the wake. Once disturbed by the passing ship, a fluid parcel on the ship's path radiates waves in all directions and at all frequencies. Wave of frequency ω spreads out radially at the phase speed of $c = g/\omega$ according to the dispersion relation. Only those parts of the waves that are stationary relative to the ship will form the ship wake, and they must satisfy the condition

$$\sigma = 0, \tag{3.5}$$

i.e.,

$$\omega = \mathbf{k} \cdot \mathbf{U}, \text{or} \quad c = \frac{\omega}{k} = \frac{\mathbf{k}}{k} \cdot \mathbf{U} \tag{3.6}$$

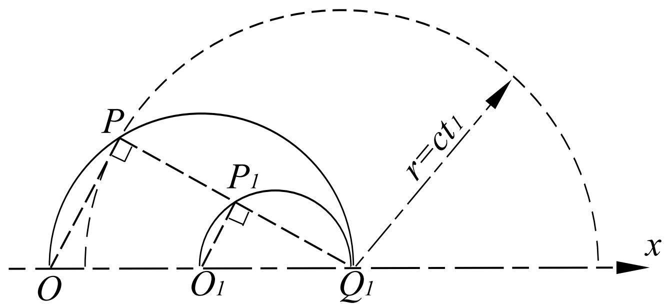

Figure 5: Waves radiated from disturbed fluid parcel

Referring to figure 5, let $O, (x = 0)$ represents the point ship in the ship-bound coordinates. The current is in the positive x direction. Any point x_1 is occupied by a fluid parcel Q_1 which was disturbed directly by the passing ship at time $t_1 = x_1/U$ earlier. This disurbed parcel radiates waves of all frequencies radially. The phase of wave at the frequency ω reaches the circle of radius ct_1 where c=g/ω by the deep water dispersion relation. Along the entire circle however only the point that satisfies (3.6) can contribute to the stationary wave pattern, as marked by P. Since $OQ_1 = x_1 = Ut_1$, $Q_1P = ct_1$ and $OP = \mathbf{Ut_1} \cdot \mathbf{k}/\mathbf{k}$, where $\mathbf{k}$ is in thedirectin of $\vec{Q_1P}$. It follows that $\triangle OPQ_1$ is a right triangle, and P lies on a semi circle with diameter OQ_1. Accounting for the radiated waves of all frequencies, hence all c, every point on the semi circle can be a part of the stationary wave phase formed by signals emitted from Q_1. Now this argument must be rectified because wave energy only travels at the group velocity which is just half of the phase velocity in deep water. Therefore stationary crests due to signals from Q_1 can only lie on the semi-circle with the diameter $O_1Q_1 = OQ_1/2$. Thus P_1 instead of P is one of the points forming a stationary crest in the ship's wake, as shown in figure 5.

Any other fluid parcel Q_2 at x_2 must have been disturbed by the passing ship at time $t_2 = x_2/U$ earlier. Its radiated signals contribute to the stationary wave pattern only along the semi circle with diameter $O_2Q_2 = OQ_2/2$. Combining the effects of all fluid parcels along the $+x$ axis, stationary wave pattern must be confined inside the wedge which envelopes all these semi circles. The half apex angle β_o of the wedge, which defines the wake, is given by

$$\sin \beta_o = \frac{Ut/4}{3Ut/4} = 1/3, \tag{3.7}$$

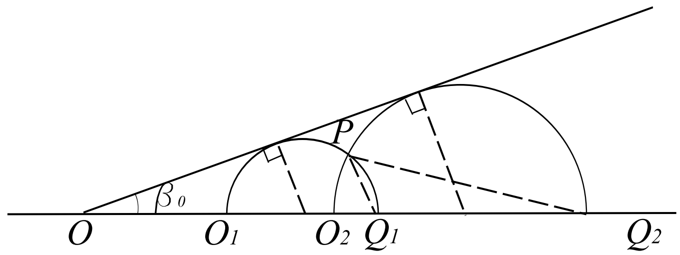

Figure 6: Wedge angle of the ship wake

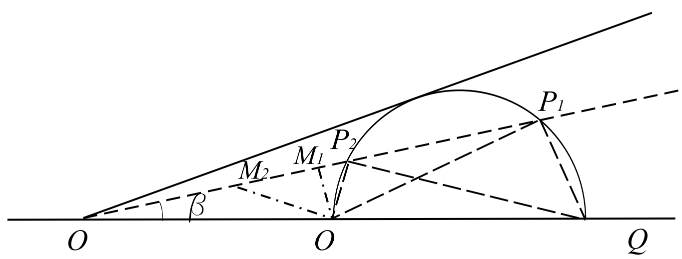

Figure 7: Geometrical relation to find Points of dependence

hence $\beta_o = \sin^{-1} 1/3 = 19.5°$, see figure 6.

Now any point P inside the wedge is on two semicircles tangent to the boundary of the wedge, i.e., there are two segments of the wave crests intersecting at P: one perpendicular to PQ_1 and one to PQ_2, as shown in figure 6.

Another way of picturing this is to examine an interior ray from the ship. Draw a semi circle with the diameter $O'Q = OQ/2$, then at the two intersections P_1 and P_2 with the ray are the two segments of the stationary wave crests. In other words, signals originated from Q contribute to the stationary wave pattern only at the two points P_1 and P_2, as shown in figure 7. Point Q can be called the point of dependence for points P_1 and P_2 on the crests.

For any interior point P there is a graphical way of finding the two points of dependence Q_1 and Q_2. Referring to figure 7, $\triangle O'QP_1$ and $\triangle O'QP_2$ are both right triangles. Draw $O_1M_1 \parallel QP_1$ and $O_2M_2 \parallel QP_2$ where M_1 and M_2 lie on the ray inclined at the angle β. it is clear that $OM_1 = OP_1/2$ and $OM_2 = OP_2/2$, and $\triangle M_1O'P_1$ and $\triangle M_2O'P_2$ are both right triangles. Hence O' lies on two semi circles with diameters M_1P_1 and

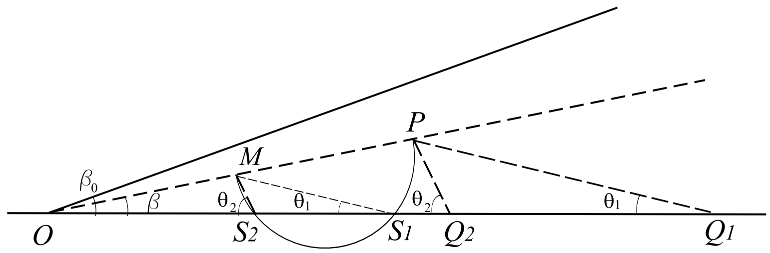

Figure 8: Points of dependence

M_2P_2.

We now reverse the process, as shown in figure 8. For any point P on an interior ray, let us mark the mid point M of OP and draw a semi circle with diameter MP. The semi circle intersects the x axis at two points S_1 and S_2. We then draw from P two lines parallel to MS_1 and MS_2, the two points of intersection Q_1 an Q_2 on the x axis are just the two points of dependence.

Let $\angle PQ_1O = \angle MS_1O = \theta_1$ and $\angle PQ_2O = \angle MS_2O = \theta_2$. then

$$\tan(\theta_i + \beta) = \frac{PS_i}{MS_i} = \frac{PS_i}{PQ_i/2} = 2\tan\theta_i \quad i = 1, 2.$$

hence

$$2\tan\theta_i = \frac{\tan\theta_i + \tan\beta}{1 - \tan\theta_i \tan\beta}$$

which is a quadratic equation for θ_i, with two solutions:

$$\left\{ \begin{array}{c} \tan\theta_1 \\ \tan\theta_2 \end{array} \right\} = \frac{1 \pm \sqrt{1 - 8\tan^2\beta}}{4\tan\beta} \tag{3.8}$$

They are real and distinct if

$$1 - 8\tan^2\beta > 0 \tag{3.9}$$

These two angles define the local stationary wave crests crossing P, and they must be perpendicular to PQ_1 and PQ_2. There are no solutions if $1 - 8\tan^2\beta < 0$, which corresponds to $\sin\beta > 1/3$ or $\beta > 19.5°$, i.e., outside the wake. At the boundary of the wake, $\beta = 19,5°$ and $\tan\beta = \sqrt{1/8}$, the two angles are equal

$$\theta_1 = \theta_2 = \tan^{-1}\frac{\sqrt{2}}{2} = 55°. \tag{3.10}$$

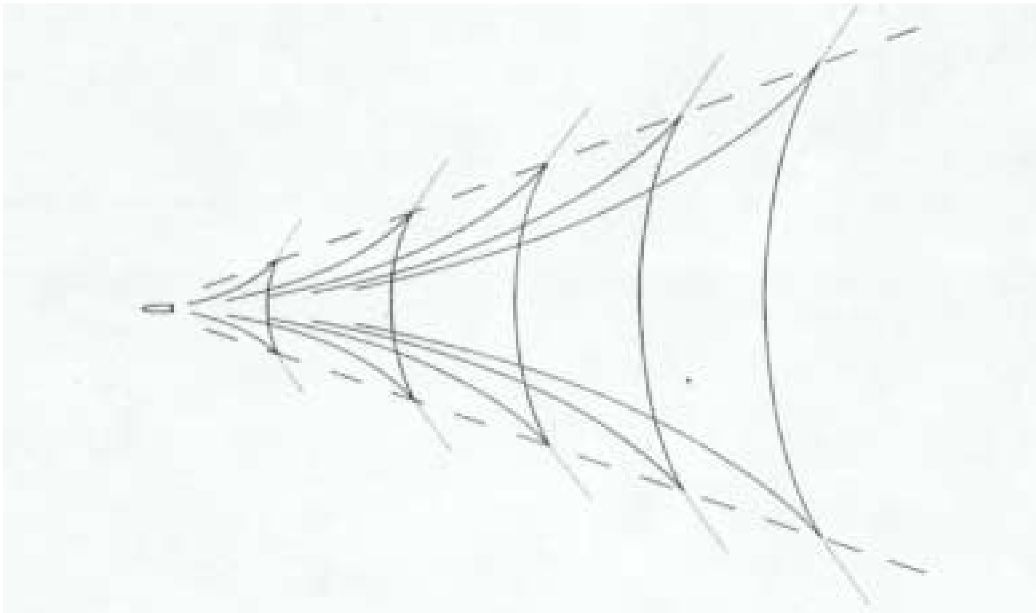

Figure 9: Diverging and transverse waves in a ship wake

By connecting these segments at all points in the wedge, one finds two systems of wave crests, the *diverging waves* and the *transverse waves*, as shown in figure div-trans.

Knowing that waves are confined in a wedge, we can estimate the behavior of the wave amplitude by balancing in order of magnitude work done by the wave drag R and the steady rate of energy flux

$$RU = (\bar{E}c_g)r \sim (|A|^2 c_g)r \tag{3.11}$$

hence

$$A \sim r^{1/2} \tag{3.12}$$

This estimate is valid throughout the wedge except near the outer boundaries, where

$$A \sim r^{-1/3} \tag{3.13}$$

by a more refined analysis (Stoker, 1957, or Wehausen & Laitone, 1960).

A beautiful photograph is shown in figure 10

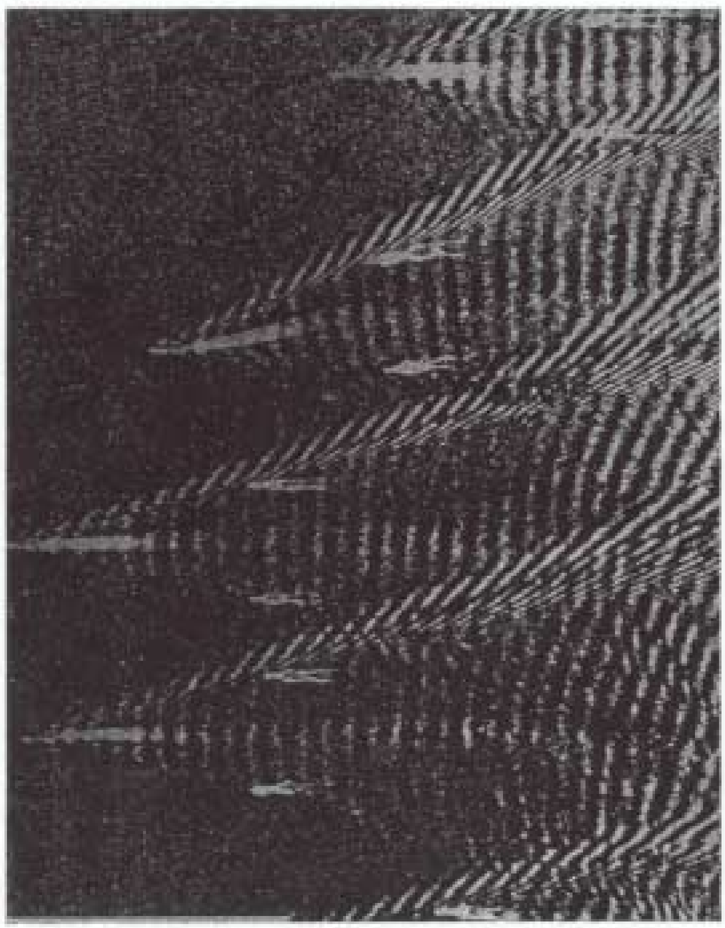

Figure 10: Ships in a straight course. From Stoker, 1957.p. 280.