I-campus project

School-wide Program on Fluid Mechanics

Modules on Waves in fluids

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CHAPTER THREE

DISPERSION OF SURFACE WATER WAVES

Except in very shallow water, one of the most outstanding physical properties of sea surface waves is *dispersion* in that waves of different wavelengths propagate at different speeds. If the waves are sufficiently steep, nonlinearity is also important. The interplay of dispersion and nonlinearity gives rise to a host of new phenomena unfamiliar in classical physics and makes surface water waves a perenial subject of fascination and challenge. In this module, only dispersion of infinitesimal waves will be discussed.

1 Progressive waves on a sea of constant depth

1.1 The velocity potential

We shall based our study on the govering equations of §1.4, and consider the simplest case of constant depth and sinusoidal waves with infinitely long crests parallel to the y axis. The motion is in the vertical plane (x, z). Let us seek a solution representing a wavetrain advancing along the x direction with frequency ω and wave number k,

$$\Phi = f(z)e^{ikx-i\omega t} \tag{1.1}$$

In order to satisfy (**??**), (2.4) and (**??**) we need

$$f'' + k^2 f = 0, \quad -h < z < 0 \tag{1.2}$$

$$-\omega^2 f + gf' + \frac{T}{\rho}k^2 f' = 0, \quad z = 0, \tag{1.3}$$

$$f' = 0, \quad z = -h \tag{1.4}$$

Clearly solution to (1.2) and (1.4) is

$$f(z) = B\cosh k(z+h)$$

implying

$$\Phi = B \cosh k(z+h) e^{ikx - i\omega t} \tag{1.5}$$

In order to satisfy (1.3) we require

$$\omega^2 = \left(gk + \frac{T}{\rho} k^3 \right) \tanh kh \tag{1.6}$$

This eigenvalue condition relates ω and k. From (**??**) we get

$$\frac{\partial \zeta}{\partial t} = \left. \frac{\partial \Phi}{\partial z} \right|_{z=0} = (Bk \sinh kh) e^{ikx - i\omega t} \tag{1.7}$$

Upon integration,

$$\zeta = A e^{ikx - i\omega t} = \frac{Bk \sinh kh}{-i\omega} e^{ikx - i\omega t} \tag{1.8}$$

where A denotes the surface wave amkplitude, it follows that

$$B = \frac{-i\omega A}{k \sinh kh}$$

and

$$\begin{aligned} \Phi &= \frac{-i\omega A}{k \sinh kh} \cosh k(z+h) e^{ikx - i\omega t} \\ &= \frac{-igA}{\omega} \left(1 + \frac{Tk^2}{g\rho} \right) \frac{\cosh k(z+h)}{\cosh kh} e^{ikx - i\omega t} \end{aligned} \tag{1.9}$$

1.2 The dispersion relation

Let us first examine the relation (1.6) between frequency and wavenumber. Here three lengths are present : the depth h, the wavelength $\lambda = 2\pi/k$, and the length $\lambda_m = 2\pi/k_m$ with

$$k_m = \sqrt{\frac{g\rho}{T}}, \quad \lambda_m = \frac{2\pi}{k_m} = 2\pi \sqrt{\frac{T}{g\rho}} \tag{1.10}$$

For reference we note that on the air-water interface, $T/\rho = 74\, cm^3/s^2, g = 980\, cm/s^2$, so that $\lambda_m = 1.73 cm$. The depth of oceanographic interest ranges from a tens of centimeters to thousand of meters. The wavelength ranges from a few centimeters to hundreds or thousands of meters.

Let us introduce

$$\omega_m^2 = 2gk_m = 2g \sqrt{\frac{g\rho}{T}} \tag{1.11}$$

then (1.6) is normalized to

$$\frac{\omega^2}{\omega_m^2} = \frac{1}{2}\frac{k}{k_m}\left(1+\frac{k^2}{k_m^2}\right)\tanh kh \tag{1.12}$$

Consider first waves of length of the order of λ_m. For depths of oceanographic interest, $h \gg \lambda$, or $kh \gg 1$, $\tanh kh \approx 1$. Hence

$$\frac{\omega^2}{\omega_m^2} = \frac{1}{2}\frac{k}{k_m}\left(1+\frac{k^2}{k_m^2}\right) \tag{1.13}$$

or, in dimensional form,

$$\omega^2 = gk + \frac{Tk^3}{\rho} \tag{1.14}$$

The phase velocity is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\left(1+\frac{Tk^2}{g\rho}\right)} \tag{1.15}$$

Defining

$$c_m = \frac{\omega_m}{k_m} \tag{1.16}$$

the preceding equation takes the normalzed form

$$\frac{c}{c_m} = \sqrt{\frac{1}{2}\left(\frac{k_m}{k}+\frac{k}{k_m}\right)} \tag{1.17}$$

Clearly

$$c \approx \sqrt{\frac{Tk}{\rho}}, \quad \text{if } \ k/k_m \gg 1, \quad \text{or} \quad \lambda/\lambda_m \ll 1 \tag{1.18}$$

Thus for wavelengths much shorter than 1.7 cm, capillarity alone is important, These are called the capillary waves. On the other hand

$$c \approx \sqrt{\frac{g}{k}}, \quad \text{if } \ k/k_m \ll 1, \quad \text{or} \quad \lambda/\lambda_m \ll 1 \tag{1.19}$$

Thus for wavelength much longer than 1.73 cm, gravity alone is important; these are called the gravity waves. Since in both limits, c beocmes large, there must be a minimum for some intermediate k. From

$$\frac{dc^2}{dk} = -\frac{g}{k^2} + \frac{T}{\rho} = 0$$

the minmum c occurs when

$$k = \sqrt{\frac{g\rho}{T}} = k_m, \quad \text{or } \lambda = \lambda_m \tag{1.20}$$

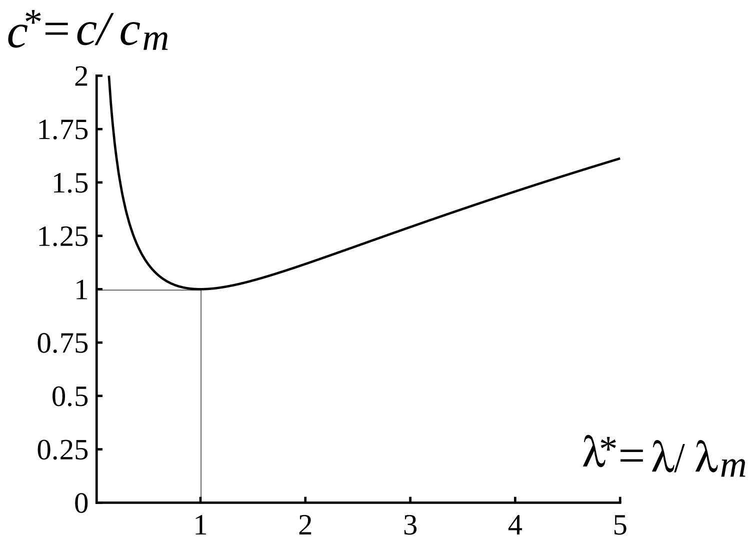

Figure 1: Phase speed of capillary-gravity waves in infinitely deep water

The smallest value of c is c_m. For the intermediate range where both capillarity and gravity are of comparble importance; the dispersion relation is plotted in figure (1).

Next we consider longer gravity waves where the depth effects are essential.

$$\omega = \sqrt{gk \tanh kh} \tag{1.21}$$

For gravity waves on deep water, $kh \gg 1$, $\tanh kh \to 1$. Hence

$$\omega \approx \sqrt{gk}, \quad c \approx \sqrt{\frac{g}{k}} \tag{1.22}$$

These are also called short gravity waves. In this category the longer waves travel faster. Any initial disturbance may be regarded as the superposition of waves of a broad spectrum of lengths. The above relation then says that waves of different lengths will eventually separate, i.e., disperse. This phenomenon is called *dispersion*, hence (1.14) or (1.15) is known as the *dispersion relation*.

If however the waves are very long in comparison to the depth so that $kh \ll 1$, then $\tanh kh \sim kh$ and

$$\omega \approx k\sqrt{gh}, \quad c \approx \sqrt{gh} \tag{1.23}$$

For intermediate values of kh, the phase speed decreases monotonically with increasing kh. All long waves with $kh \ll 1$ travel at the same maximum speed limited by the

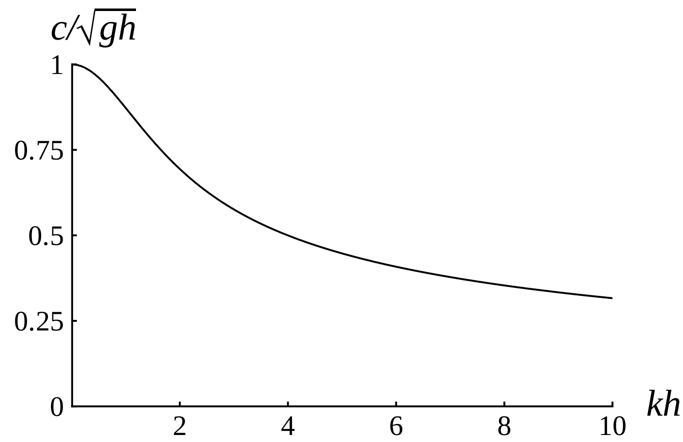

Figure 2: Phase speed of capillary-gravity waves in water of finite depth

depth, $\sqrt{gh}$, hence they are non-dispersive. The dispersion relation is plotted in figure (2) for a wide range of wavelengths.

1.3 The flow field

For arbitrary k/k_m and kh, the velocities and dynamic pressure are easily found from the potential (1.9) as follows

$$u = \frac{\partial \Phi}{\partial x} = \frac{gkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh}e^{ikx-i\omega t} \tag{1.24}$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{-igkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z+h)}{\cosh kh}e^{ikx-i\omega t} \tag{1.25}$$

$$p = -\rho\frac{\partial \Phi}{\partial t} = \rho g A\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh}e^{ikx-i\omega t} \tag{1.26}$$

Note that all these quantities decay monotonically in depth.

In deep water, $kh \gg 1$,

$$u = \frac{gkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \tag{1.27}$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{-igkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \tag{1.28}$$

$$p = -\rho\frac{\partial \Phi}{\partial t} = \rho g A\left(1+\frac{Tk^2}{g\rho}\right)e^{kz}e^{ikx-i\omega t} \tag{1.29}$$

All dynamical quantities diminish exponentially to zero as $kz \to -\infty$. Thus the fluid motion is limited to the surface layer of depth $O(\lambda)$. Gravity and capillary-gravity waves are therefore surface waves.

For pure gravity waves in shallow water, $T = 0$ and $kh \ll 1$, we get

$$u = \frac{gkA}{\omega} e^{ikx - i\omega t} \tag{1.30}$$

$$w = 0, \tag{1.31}$$

$$p = -\rho \frac{\partial \Phi}{\partial t} = \rho g A e^{ikx - i\omega t} = \rho g \zeta \tag{1.32}$$

Note that the horizontal velocity is uniform in depth while the vertical velocity is negligible. Thus the fluid motion is essentially horizontal. The total pressure

$$P = p_o + p = \rho g(\zeta - z) \tag{1.33}$$

is hydrostatic and increases linearly with depth from the free surface.

1.4 The particle orbit

In fluid mechanics there are two ways of describing fluid motion. In the Lagrangian scheme, one follows the trajectory x, z of all fluid particles as functions of time. Each fluid particle is identified by its static or intital position x_o, z_o. Therefore the instantaneous position at time t depends parametrically on x_o, z_o. In the Eulerian scheme, the fluid motion at any instant t is described by the velocity field at all fixed positions x, z. As the fluid moves, the point x, z is occupied by different fluid particles at different times. At a particular time t, a fluid particle originally at (x_o, z_o) arrives at x, z, hence its particle velocity must coincide with the fluid velocity there,

$$\frac{dx}{dt} = u(x, z, t), \quad \frac{dz}{dt} = w(x, z, t) \tag{1.34}$$

Once u, w are known for all x, z, t, we can in principle integrate the above equations to get the particle trajectory. This Euler-Lagrange problem is in general very difficult.

In small amplitude waves, the fluid particle oscillates about its mean or initial position by a small distance. Integration of (1.34) is relatively easy. Let

$$x(x_o, z_o, t) = x_o + x'(x_o, z_o, t), \quad \text{and} z(x_o, z_o, t) = z_o + x'(x_o, z_o, t) \tag{1.35}$$

then $x' \ll x, z' \ll z$ in general. Equation (1.34) can be approximated by

$$\frac{dx'}{dt} = u(x_o, z_o, t), \qquad \frac{dz'}{dt} = w(x_o, z_o, t) \tag{1.36}$$

From (1.24) and (1.25), we get by integration,

$$\begin{aligned} x' &= \frac{gkA}{-i\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o+h)}{\cosh kh}e^{ikx_o - i\omega t} \\ &= -\frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o+h)}{\cosh kh}\sin(kx_o - \omega t) \end{aligned} \tag{1.37}$$

$$\tag{1.38}$$

$$\begin{aligned} z' &= \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o+h)}{\cosh kh}e^{ikx_o - i\omega t} \\ &= \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o+h)}{\cosh kh}\cos(kx_o - \omega t) \end{aligned} \tag{1.39}$$

$$\tag{1.40}$$

Letting

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{gkA}{\omega^2 \cosh kh}\left(1+\frac{Tk^2}{g\rho}\right)\begin{pmatrix} \cosh k(z_o+h) \\ \sinh k(z_o+h) \end{pmatrix} \tag{1.41}$$

we get

$$\frac{x'^2}{a^2} + \frac{z'^2}{b^2} = 1 \tag{1.42}$$

The particle trajectory at any depth is an ellipse. Both horizontal (major) and vertical (minor) axes of the ellipse decrease monotonically in depth. The minor axis diminshes to zero at the seaebed, hence the ellipse collapses to a horizontal line segment. In deep water, the major and minor axes are equal

$$a = b = \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)e^{kz_o}, \tag{1.43}$$

therefore the orbits are circles with the radius diminishing exponentially with depth.

Also we can rewrite the trajectory as

$$x' = \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z_o+h)}{\cosh kh}\sin(\omega t - kx_o) \tag{1.44}$$

$$z' = \frac{gkA}{\omega^2}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z_o+h)}{\cosh kh}\sin(\omega t - kx_o - \frac{\pi}{2}) \tag{1.45}$$

When $\omega t - kx_o = 0$, $x' = 0$ and $z' = b$. A quarter period later, $\omega t - k_o = \pi/2$, $x' = a$ and $z' = 0$. Hence as time passes, the particle traces the ellptical orbit in the clockwise direction.

1.5 Energy and Energy transport

Beneath a unit length of the free surface, the time-averaged kinetic energy density is

$$\bar{E}_k = \frac{\rho}{2}\int_{-h}^{0} dz\left(\overline{u^2}+\overline{w^2}\right) \tag{1.46}$$

whereas the instantaneous potential energydensity is

$$E_p = \frac{1}{2}\rho g\zeta^2 + T\frac{(ds-dx)}{dx} = \frac{1}{2}\rho g\zeta^2 + T\left(\sqrt{1+\zeta_x^2}-1\right) = \frac{1}{2}\rho g\zeta^2 + T\zeta_x^2 \tag{1.47}$$

Hence the time-average is

$$\bar{E}_p = \frac{1}{2}\rho g\overline{\zeta^2} + \frac{T}{2}\overline{\zeta_x^2} \tag{1.48}$$

Let us rewrite (1.24) and (1.25) in (1.48):

$$u = \Re\left\{\frac{gkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\cosh k(z+h)}{\cosh kh}e^{ikx}\right\}e^{-i\omega t} \tag{1.49}$$

$$w = \Re\left\{\frac{-igkA}{\omega}\left(1+\frac{Tk^2}{g\rho}\right)\frac{\sinh k(z+h)}{\cosh kh}e^{ikx}\right\}e^{-i\omega t} \tag{1.50}$$

Then

$$\begin{aligned}
\bar{E}_k &= \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{1}{\cosh^2 kh}\int_{-h}^{0} dz\left[\cosh^2 k(z+h)+\sinh^2 k(z+h)\right] \\
&= \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{\sinh 2kh}{2k\cosh^2 kh} = \frac{\rho}{4}\left(\frac{gkA}{\omega}\right)^2\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{\sinh kh}{k\cosh kh} \\
&= \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{g\rho}\right)^2\frac{gk\tanh kh}{\omega^2} = \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{g\rho}\right)
\end{aligned} \tag{1.51}$$

after using the dispersion relation. On the other hand,

$$\bar{E}_p = \frac{\rho g A^2}{4}\left(1+\frac{Tk^2}{\rho g}\right) \tag{1.52}$$

Hence the total enegy density is

$$\bar{E} = \bar{E}_k + \bar{E}_p = \frac{\rho g A^2}{2}\left(1+\frac{Tk^2}{\rho g}\right) = \frac{\rho g A^2}{2}\left(1+\frac{k^2}{k_m^2}\right) = \frac{\rho g A^2}{2}\left(1+\frac{\lambda_m^2}{\lambda^2}\right) \tag{1.53}$$

Note that the total energy is equally divided between kinetic and potential energies; this is called the equipartition of energy.

We leave it as an exercise to show that the power flux (rate of energy flux) across a station x is

$$\frac{d\bar{E}}{dt} = \int_{-h}^{0} \overline{pu}\, dz - T\overline{\zeta_x \zeta_t} = -\rho \int_{-h}^{0} \overline{\Phi_t \Phi_x}\, dz - T\overline{\zeta_x \zeta_t} = \bar{E} c_g \tag{1.54}$$

where c_g is the speed of energy transport , or the group velocity

$$c_g = \frac{d\omega}{dk} = \frac{c}{2}\left\{\frac{\frac{k_m^2}{k^2}+3}{\frac{k_m^2}{k^2}+1} + \frac{2kh}{\sinh 2kh}\right\} = \frac{c}{2}\left\{\frac{\frac{\lambda^2}{\lambda_m^2}+3}{\frac{\lambda^2}{\lambda_m^2}+1} + \frac{2kh}{\sinh 2kh}\right\} \tag{1.55}$$

For pure gravity waves, $k/k_m \ll 1$ so that

$$c_g = \frac{c}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{1.56}$$

where the phase velocity is

$$c = \sqrt{\frac{g}{k}\tanh kh} \tag{1.57}$$

In very deep water $kh \gg 1$, we have

$$c_g = \frac{c}{2} = \frac{1}{2}\sqrt{\frac{g}{k}} \tag{1.58}$$

The shorter the waves the smaller the phase and group velocities. In shallow water $kh \ll 1$,

$$c_g = c = \sqrt{gh} \tag{1.59}$$

Long waves are the fastest and no longer dispersive.

For capillary-gravity waves with $kh \gg 1$, we have

$$c_g = \frac{c}{2}\left\{\frac{\frac{k_m^2}{k^2}+3}{\frac{k_m^2}{k^2}+1}\right\} = \frac{c}{2}\left\{\frac{\frac{\lambda^2}{\lambda_m^2}+3}{\frac{\lambda^2}{\lambda_m^2}+1}\right\}, \quad k_m = \frac{2\pi}{\lambda_m}\sqrt{\frac{\rho g}{T}} \tag{1.60}$$

where

$$c = \sqrt{\frac{g}{k} + \frac{Tk^3}{\rho}} \tag{1.61}$$

Note that $c_g = c$ when $k = k_m$, and

$$c_g \gtrless c, \quad \text{if} \quad k \gtrless k_m \tag{1.62}$$

In the limit of pure capillary waves of $k \gg k_m$, $c_g = 3c/2$. For pure gravity waves $c_g = c/2$ as in (1.58).

2 Dispersion of transient disturbance

The solution for monochromatic waves already suggests that waves of different wavelengths disperse by travelling at different velocities. Let us examine in more detail the consequence of an initial disturbance which is represented by the sum of infinitely many sinusoids with a wide spectrum. To this end we shall employ the tool of Fourier transform.

Let us consider two dimensional capillary-gravity waves in very deep water. Recall from §1.4 for convenience that the velocity potential satisfies the Laplace equation

$$\Phi_{xx} + \Phi_{zz} = 0, \quad -\infty < z < 0. \tag{2.1}$$

On the free surface the dynamical boundary condition requires

$$-\rho g \zeta - \rho \frac{\partial \Phi}{\partial t} + T \frac{\partial^2 \zeta}{\partial x^2} = 0, \quad z = 0. \tag{2.2}$$

The kinematic condition requires

$$\zeta_t = \Phi_z, \quad z = 0. \tag{2.3}$$

Combination of the two yields

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} - \frac{T}{\rho} \frac{\partial^3 \Phi}{\partial x^2 \partial z} = 0, \quad z = 0 \tag{2.4}$$

At great depth, the velocity vanishes

$$\Phi_x, \Phi_z \to 0, \quad z \to -\infty. \tag{2.5}$$

Since conditions (2.3) and (2.2) involve first-order time derivatives, we must prescribe the initial data for $\Phi(x,0,0)$ and $\zeta(x,0)$ on the free surface. Physically $\Phi(x,0,0)$ is equivalent to an impulsive pressure applied on the free surface. Here we shall only illustrate the effects of a prescribed initial displacement of the free surface,

$$\Phi(x,0,0) = 0, \quad \zeta(x,0) = \zeta_o(x) = \text{given}. \tag{2.6}$$

2.1 Solution by Fourier transform

Let us define the Fourier transform of $f(x)$ and its inverse $\overline{f}(k)$ by

$$\bar{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} \bar{f}(k)\, dk \tag{2.7}$$

Then it is possible to show that the solution for the surface displacement is

$$\zeta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \overline{\zeta}_o(k) \cos\omega t \tag{2.8}$$

where $\bar{\zeta}(k,t)$ is the Fourier transform of $\zeta(x,0)$, and the potential is

$$\Phi(k,z,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \overline{\zeta}_o(k) \left(g + \frac{Tk^2}{\rho}\right) \frac{\sin\omega t}{\omega} e^{|k|z} \tag{2.9}$$

with

$$\omega = \left[|k|\left(g + \frac{Tk^2}{\rho}\right)\right]^{1/2} \tag{2.10}$$

Detailed derivation is as follows.

The transform of Laplace equation is

$$\overline{\Phi}_{zz} - k^2\overline{\Phi} = 0, \quad z < 0 \tag{2.11}$$

From the combined free surface condition, we get

$$\overline{\Phi}_{tt} + \left(g + \frac{Tk^2}{\rho}\right)\overline{\Phi}_z = 0 \tag{2.12}$$

From the dynamical condition on the free surface

$$\overline{\Phi}_t = -\left(g + \frac{Tk^2}{\rho}\right)\overline{\zeta} \tag{2.13}$$

At great depth

$$\overline{\Phi}, \overline{\Phi}_z \to 0, \quad z \to -\infty \tag{2.14}$$

The initial conditions on the free surface are

$$\overline{\zeta}(k,0) = \overline{\zeta}_o(k) \tag{2.15}$$

$$\overline{\Phi}(k,0,0) = 0 \tag{2.16}$$

The solution of (2.11) is

$$\overline{\Phi}(k,z,t) = A(k,t)e^{|k|z}$$

From (2.12), A must satisfy

$$A_{tt} + |k|\left(g + \frac{Tk^2}{\rho}\right)A = 0, \quad t > 0$$

From (2.16) and (2.15), the initial conditions for A are

$$A(k,0) = 0,$$
$$A_t(k,0) = -\overline{\zeta}(k,0)\left(g + \frac{Tk^2}{\rho}\right) = -\overline{\zeta}_o(k)\left(g + \frac{Tk^2}{\rho}\right)$$

Hence

$$A = -\overline{\zeta}_o(k)\left(g + \frac{Tk^2}{\rho}\right)\frac{\sin \omega t}{\omega}$$

The solution for the tranformed potential is

$$\overline{\Phi}(k,z,t) = -\overline{\zeta}_o(k)\left(g + \frac{Tk^2}{\rho}\right)\frac{\sin \omega t}{\omega}e^{|k|z} \tag{2.17}$$

The transform of the surface displacement is

$$\overline{\zeta}(x,t) = -\frac{\overline{\Phi}_t(k,0,t)}{g + \frac{Tk^2}{\rho}} = \overline{\zeta}_o(k)\cos \omega t \tag{2.18}$$

By Fourier inversion the solutions are given by (2.8) and (2.9).

To be concrete we shall take

$$\zeta_o(x) = \frac{Sb}{\pi(x^3 + b^2}$$

which is a hump of area S; the Fourier transform is

$$\overline{\zeta}_o(k) = Se^{-|k|b}$$

which is even in k. It follows that

$$\zeta(x,t) = \frac{S}{\pi}\int_0^\infty dke^{-kb}\cos kx \cos \omega t$$

which can be manipulated to

$$\zeta(x,t) = \frac{S}{2\pi}\Re\int_0^\infty dke^{-kb}\left(e^{ikx-i\omega t} + e^{ikx+i\omega t}\right)dk \tag{2.19}$$

The first term in the integrand represents the right-going wave while the second, left-going. Each part correponds to a superposition of sinusoidal wave trains over the entire range of wave numbers, within the small range $(k, k+dk)$ the spectral amplitude is Se^{-kb}. In general explicit evaluation of the Fourier integrals is not feasible. We shall therefore only seek approximate information.

2.2 Method of stationary phase

The method of stationary phase is particularly useful for asymptotic approximation of Fourier integrals,

$$I(t) = \int_a^b F(k)e^{itf(k)}\,dk \tag{2.20}$$

for real f and very large t. Let us first give a quick derivation of the mathematical result.

If t is large, then as k increases along the path of integration both the real and imaginary parts of the exponential function

$$\cos(tf(k)) + i\sin(tf(k))$$

oscillates rapidly between -1 to +1, resulting in cancellations unless there is a point of stationary phase k_o within (a,b) so that

$$\frac{df(k_o)}{dk} = f'(k_o) = 0, \quad a < k_o < b. \tag{2.21}$$

Then

$$f(k) = f(k_o) + \frac{1}{2}(k-k_o)^2 f''(k_o) + \cdots$$

and

$$\begin{aligned} e^{itf(k)} &\approx e^{itf(k_o)}\exp\left[it(f(k)-f(k_o))\right] \\ &\approx e^{itf(k_o)}\left\{\cos\left[t(f(k)-f(k_o))\right] + i\sin\left[t(f(k)-f(k_o))\right]\right\} \end{aligned}$$

As sketched in Figure 3, contribution to the Fourier integral is dominated by the cosine part in the neighborhood of k_o. The integral can be approximated by

$$I(t) \approx F(k_o)e^{itf(k_o)}\int_a^b \exp\left(\frac{it}{2}(k-k_o)^2 f''(k_o)\right)dk$$

With an error of $O(1/t)$, we also replace the limits of the last integral by $\pm\infty$; the justification is omitted here. Now it is known that

$$\int_{-\infty}^{\infty} e^{\pm itk^2}\,dk = \sqrt{\frac{\pi}{t}}e^{\pm i\pi/4}$$

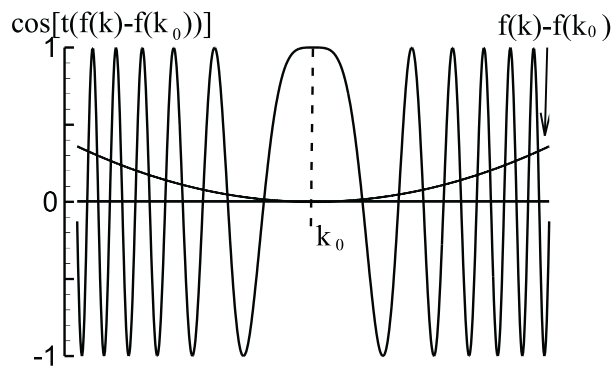

Figure 3: Neighborhood of the point of stationary phase

It follows that

$$I(t) = \int_a^b F(k)e^{itf(k)}\,dk \approx F(k_o)e^{itf(k_o)\pm i\pi/4}\left[\frac{2\pi}{t|f''(k_o)|}\right]^{1/2} + O\left(\frac{1}{t}\right), \quad \text{if } \; k_o \in (a,b), \tag{2.22}$$

where the sign is $+$ (or $-$) if $f''(k_o)$ is positive (or negative). It can be shown that if ther is no stationary point in the range(a,b), then the integral $I(t)$ is small

$$I(t) = O\left(\frac{1}{t}\right), \quad \text{if } \; k_o \notin (a,b). \tag{2.23}$$

2.3 Wave dispersion at large x and t.

Let us apply this result to the right-going wave

$$\zeta_+(x,t) = \frac{1}{2\pi}\Re\int_0^\infty dk\bar{\zeta}(k,0)e^{it(kx/t-i\omega)}dk \tag{2.24}$$

where

$$\omega = \left[|k|\left(g + \frac{Tk^2}{\rho}\right)\right]^{1/2} \tag{2.25}$$

For an observer travelling at a given speed, x/t =constant. We have

$$f(k) = kx/t - \omega(k), \tag{2.26}$$

There is a stationary point k_o at the root of

$$x/t = \omega'(k) = c_g = \frac{g + \frac{3Tk^2}{\rho}}{2\left(gk + \frac{Tk^3}{\rho}\right)^{1/2}} \tag{2.27}$$

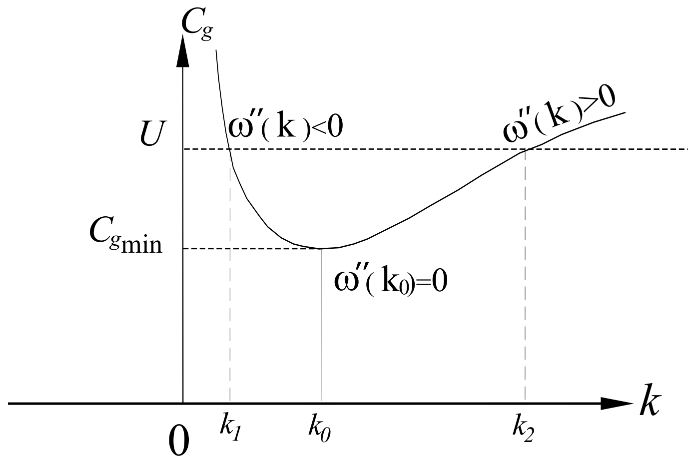

Figure 4: Group velocity as function of k of capillary-gravity waves in deep water

which is plotted in Figure 4. Note that $c_g(k)$ is large for both small and large k :

$$c_g \sim \frac{1}{2}\sqrt{\frac{g}{k}}, \quad \text{for small } k,$$

and

$$c_g \sim \frac{1}{2}\sqrt{\frac{Tk}{\rho}}, \quad \text{for large } k,$$

Hence there is a minimum c_g occuring at k_o where $\omega''(k_o) = 0$.

For any speed $x/t >$ min c_g, eq. (2.27) has two roots (stationary points). At the smaller root $k_1 < k_o$, $\omega''(k_o) < 0$; at the larger one $k_2 > k_o$, $\omega''(k_o) > 0$. Adding the contributions from both we get the final result for the right-going wave

$$\begin{aligned} \zeta_+(x,t) \quad &\sim \quad \frac{1}{2\pi}e^{-k_1 b}\sqrt{\frac{2\pi}{t|\omega''(k_1)|}}\cos\left(k_1 x - \omega(k_1)t - \pi/4\right) \\ &+ \quad \frac{1}{2\pi}e^{-k_2 b}\sqrt{\frac{2\pi}{t|\omega''(k_2)|}}\cos\left(k_2 x - \omega(k_2)t + \pi/4\right) \end{aligned} \tag{2.28}$$

Physically an observer travelling at the speed x/t sees two trains of simple harmonic waves with wavenumbers k_1 and k_2, corresponding respectively to gravity (longer) and capillary (shorter) waves. The local wavelengths are such that their group velocities match the speed of the observer. The faster the observer, the shorter the capillary waves and the longer the gravity waves. If a snapshot is taken for all $x > 0$, then the longer gravity waves are at the very front, followed by shorter and shorter gravity waves. However the shortest capillary waves lead the longer ones, see figure 5. Because e^{-kb}

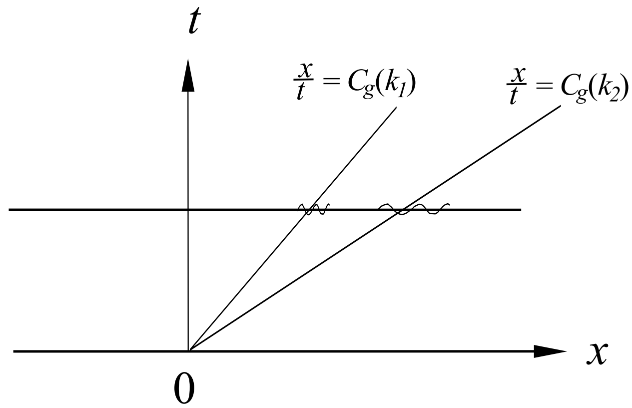

Figure 5: Large-time dispersion of capillary-gravity waves in deep water

is is the greatest at $k = 0$, the longest waves are the biggest. The entire disturbance attenuates in time as $t^{-1/2}$.

Note also that for $x/t \approx \min c_g$, the second derivative $f''(k_o) = -\omega''(k_o) = 0$. Hence the asymptotic formula breaks down. A better approximation is needed, and is left as an exercise.

Homework Show by expanding

$$\omega(k) = \omega(k_o) + (k - k_o)\omega'(k_o) + \frac{1}{6}(k - k_o)^3\omega'''(k_o) + \cdots \tag{2.29}$$

that for large x and t,

$$\zeta_+(x,t) \approx \left(\frac{2}{t\omega''(k_o)}\right)^{1/3} e^{-k_o b} A_i(Z)\cos(\omega(k_o) - k_o x) \tag{2.30}$$

where Ai(Z) is the Airy function with the argument

$$Z = \left(\frac{2}{\omega'''(k_o)t}\right)^{1/3} (c_g(k_o)t - x))$$

It can be defined by the integral

$$Ai(-Z) = \frac{1}{\pi}\int_0^\infty \cos\left(-Z\alpha + \frac{\alpha^3}{3}\right) d\alpha \tag{2.31}$$

and is related to Bessel function of order $\pm 1/3$,

$$Ai(-Z) = \frac{Z^{1/2}}{3}\left[J_{\frac{1}{3}}\left(\frac{2\sqrt{Z}}{3}Z^{3/2}\right) + J_{-\frac{1}{3}}\left(\frac{2\sqrt{(Z)}}{3}Z^{3/2}\right)\right] \tag{2.32}$$

Discuss the physical picture.

2.4 Energy propagation

Finally we examine the propagation of wave energy in this transient problem. It sufficies to examine the gravity wave part. Using (2.28) the local energy density of the gravity wave is:

$$E = \frac{1}{2}\rho\omega^2|\mathcal{A}|^2 = \frac{\rho\omega^2}{8\pi t}\frac{\bar{\zeta}(k_1)^2}{(\omega''(k_1))^2}$$

At any given t, the waves between two observers moving at slightly different speeds, $c_g(k_1')$ and $c_g(k_1'')$, i.e., between two rays $x/t = c_g(k_1')$ and $x/t = c_g(k_1'')$ are essentially simple harmonic so that the total energy is

$$\int_{x_1}^{x_2} dxE = \int_{x_1}^{x_2} dx\frac{\rho\omega^2}{8\pi t}\frac{(\bar{\zeta}(k_o))^2}{\omega''(k_1)}$$

Since $x = \omega'(k)t$ for fixed t, we have

$$\frac{dx}{t} = \omega''(k_1)dk_1$$

Now for $x_2 > x_1$, $k_1'' > k_1'$, it follows that

$$\int_{x_1}^{x_2} dxE = \int_{k_1'}^{k_1''} dk_1\frac{\rho\omega^2}{8\pi}(\bar{\zeta}(k_1))^2 = \text{constant}$$

Therefore the total energy between two observers moving at the local group velocity remains the same for all time. In other words, waves are transported by the local group velocity even in transient dispersion.

3 Narrow-banded dispersive waves in general

In this section let us discuss the superposition of progressive sinusoidal waves with the amplitudes spread over a narrow spectrum of wave numbers

$$\zeta(x,t) = \int_0^\infty |\mathcal{A}(k)|\cos(kx - \omega t - \theta_A)dk = \Re\int_0^\infty \mathcal{A}(k)e^{ikx - i\omega t}dk \tag{3.1}$$

where$\mathcal{A}(k)$ is complex denotes the dimensionless amplitude spectrum of dimension $(\text{length})^2$. The component waves are dispersive with a general nonlinear relation $\omega(k)$. Let $\mathcal{A}(k)$ be different from zero only within a narrow band of wave numbers centered at k_o. Thus the integrand is of significance only in a small neighborhood of k_o. We then approximate the integral by expanding for small $\Delta k = k - k_o$ and denote $\omega_o = \omega(k_o)$, $\omega'_o = \omega'(k_o)$, and $\omega''_o = \omega''(k_o)$,

$$\begin{aligned}
\zeta &= \Re\left\{e^{ik_o x - i\omega_o t}\int_0^\infty \mathcal{A}(k)\, e^{i\Delta k x - i(\omega-\omega_o)t} dk\right\} \\
&= \Re\left\{e^{ik_o x - i\omega_o t}\int_0^\infty dk\, \mathcal{A}(k) \exp\left[i\Delta k x - i\left(\omega'_o \Delta k + \frac{1}{2}\omega''_o(\Delta k)^2\right)t + \cdots\right]\right\} \\
&= \Re\left\{A(x,t) e^{ik_o x - i\omega_o t}\right\} \qquad (3.2)
\end{aligned}$$

where

$$A(x,t) = \int_0^\infty dk\, \mathcal{A}(k)\, \exp\left[i\Delta k x - i\left(\omega'_o \Delta k + \frac{1}{2}\omega''_o(\Delta k)^2\right)t + \cdots\right] \qquad (3.3)$$

Although the integration is formally extends from 0 to ∞, the effective range is only from $k_o - (\Delta k)_m$ to $k_o + (\Delta k)_m$, i,.e., the total range is $O((\Delta k)_m)$, where $(\Delta k)_m$ is the bandwidth. Thus the total wave is almost a sinusoidal wavetrain with frequency ω_o and wave number k_o, and amplitude $A(x,t)$ whose local value is slowly varying in space and time. $A(x,t)$ is also called the envelope. How slow is its variation?

If we ignore terms of $(\Delta k)^2$ in the integrand, (3.3) reduces to

$$A(x,t) = \int_0^\infty dk\, \mathcal{A}(k)\, \exp\left[i\Delta k(x - \omega'_o t)\right] \qquad (3.4)$$

Clearly $A = A(x - \omega'_o t)$. Thus the envelope itself is a wave traveling at the speed ω'_o. This speed is called the group velocity,

$$c_g(k_o) = \left.\frac{d\omega}{dk}\right|_{k_o} \qquad (3.5)$$

Note that the characteristic length and time scales are $(\Delta k_m)^{-1}$ and $(\omega'_o \Delta k_m)^{-1}$ respectively, therefore much longer than those of the component waves : k_o^{-1} and ω_o^{-1}. In other words, (3.3) is adequate for the slow variation of A_e in the spatial range of $\Delta k_m\, x = O(1)$ and the time range of $\omega'_o \Delta k_m\, t = O(1)$.

As a specific example we let the amplitude spectrum be a real constant within the narrow band of $k_o - \kappa, k_o + \kappa$,

$$\zeta = \mathcal{A}\int_{k_o-\kappa}^{k_o+\kappa} e^{ikx - i\omega(k)t} dk, \quad \kappa \ll k_o \qquad (3.6)$$

then

$$\begin{aligned}
\zeta &= k_o \mathcal{A} e^{ik_o x - i\omega_o t}\int_{-\kappa}^{\kappa} d\xi e^{ik_o\xi(x - c_g t)} + \cdots \\
&= \frac{2\mathcal{A}\sin\kappa(x - c_g t)}{x - c_g t} e^{ik_o x - i\omega_o t} = A e^{ik_o x - i\omega_o t} \qquad (3.7)
\end{aligned}$$

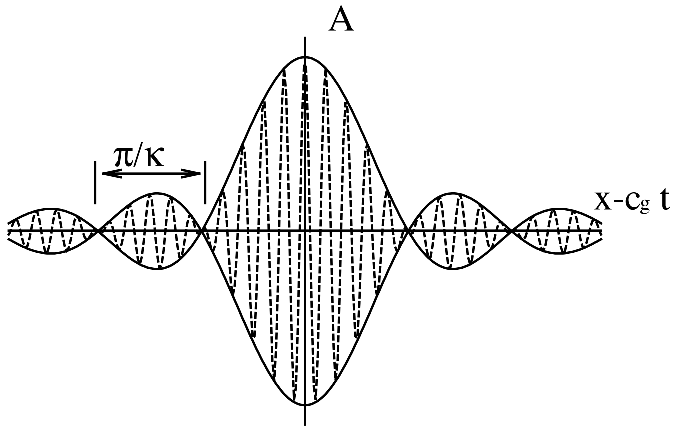

Figure 6: Envelope of waves with a rectangular band of wavenumbers

where $\xi = k - k_o/k_o$ and

$$A = \frac{2\mathcal{A}\,\sin\kappa(x - c_g t)}{(x - c_g t)} \tag{3.8}$$

as plotted in figure (6).

By differentiation, it can be verified that

$$\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x} = 0, \tag{3.9}$$

Multiplying (3.9) by A^*,

$$A^* \frac{\partial A}{\partial t} + c_g A^* \frac{\partial A}{\partial x} = 0,$$

and adding the result to its complex conjugate,

$$A \frac{\partial A^*}{\partial t} + c_g A \frac{\partial A^*}{\partial x} = 0,$$

we get

$$\frac{\partial |A|^2}{\partial t} + c_g \frac{\partial |A|^2}{\partial x} = 0 \tag{3.10}$$

We have seen that for a monochromatic wave train the energy density is proportional to $|A|^2$. Thus the time rate of change of the *local* energy density is balanced by the net flux of energy by the group velocity.

Now let us examine the more accurate approximation (3.3). By straightforward differentiation, we find

$$\frac{\partial A}{\partial t} = \int_0^\infty \left[-i\omega'(k_o)\Delta k - \frac{i\omega''(k_o)}{2}(\Delta k)^2 \right] \mathcal{A}(k) e^{iS} dk$$

$$\frac{\partial A}{\partial x} = \int_0^\infty (i\Delta k)\mathcal{A}(k)e^{iS}dk$$
$$\frac{\partial^2 A}{\partial x^2} = \int_0^\infty \left(-(\Delta k)^2\right)\mathcal{A}(k)e^{iS}dk$$

where

$$S = \Delta k\, x - \omega_o'\Delta k\, t - \frac{1}{2}\omega_o''(\Delta k)^2\, t \tag{3.11}$$

is the phase function. It can be easily verified that

$$\frac{\partial A}{\partial t} + \omega_o'\frac{\partial A}{\partial x} = \frac{i\omega_o''}{2}\frac{\partial^2 A}{\partial x^2} \tag{3.12}$$

By keeping the quadratic term in the expansion, (3.12) is now valid for a larger spatial range of $(\Delta k)^2 x = O(1)$. In the coordinate system moving at the group velocity, $\xi = x - c_g t, \tau = t$, we easily find

$$\frac{\partial A(\xi,\tau)}{\partial t} = \frac{\partial A}{\partial \tau} - c_g\frac{\partial A}{\partial \xi}, \quad \frac{\partial A(\xi,\tau)}{\partial x} = \frac{\partial A}{\partial x}$$

so that (3.12) simplifies to the Schrödinger equation:

$$\frac{\partial A}{\partial \tau} = \frac{i\omega_o''}{2}\frac{\partial^2 A}{\partial \xi^2} \tag{3.13}$$

By manipulations similar to those leading to (3.10), we get

$$\frac{\partial |A|^2}{\partial \tau} = \frac{i\omega_o''}{2}\frac{\partial}{\partial \xi}\left(A^*\frac{\partial A}{\partial \xi} - A\frac{\partial A^*}{\partial \xi}\right) \tag{3.14}$$

Thus the local energy density is not conserved over a long distance of propagation. Higher order effects of dispersion redistribute energy to other parts of the envelope. For either a wave packet whose envelope has a finite length ($A(\pm\infty) = 0$), or for a periodically modulated envelope ($A(x) = A(x+L)$), we can integrate (3.14) to give

$$\frac{\partial}{\partial \tau}\int |A|^2 d\xi = 0 \tag{3.15}$$

where the integration extends over the entire wave packet or the group period. Thus the total energy in the entire wave packet or in a group period is conserved.