I-campus project

School-wide Program on Fluid Mechanics

MODULE ON WAVES IN FLUIDS

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Chapter 1. SAMPLE WAVE PROBLEMS

To describe a problem in mathematical terms, one must make use of the basic laws that govern the elements of the problem. In continuum mechanics, these are the conservation laws for mass and momentum. In addition, empirical constitutive laws are often needed to relate certain unknown variables such as the equations of state, and relations between stress and strain rates, etc.

To derive the conservation law one may consider an infinitesimal element (a line segment, area or volume element), yielding a differential equation directly. Alternately, one may consider a control volume (or area, or line segment) of arbitrary size in the medium of interest. The law is first obtained in integral form; a differential equation is then derived by using the arbitrariness of the control volume. The two approaches are completely equivalent.

1 Wave propagation in arteries

Let us first examine the pulsating flow of blood in an artery whose wall is thin and elastic. As a first exercise let us assume that there is only pulsation but no net flow. Because of the pressure gradient in the blood, the artery wall must deform. The elastic restoring force in the wall makes it possible for waves to propagate.

The artery radius $a(x,t)$ varies from the constant mean a_o in time and along the artery (in x). Let the local cross sectional area be $S = \pi a^2$, and the averaged velocity be $u(x,t)$. Consider a fixed geometrical volume between x and $x + dx$, through which fluid moves in and out. Conservation of mass requires

$$\frac{\partial S}{\partial t} + \frac{\partial (uS)}{\partial x} = 0, \tag{1.1}$$

Next the momentum balance. The time rate of momentum change in the volume must be balanced by the net influx of momentum through the two ends and the pressure force acting on all sides. The rate of mometum change is

$$\frac{\partial(\rho u S)}{\partial t} \tag{1.2}$$

The net rate of momentum influx is

$$-\frac{\partial(\rho u^2 S)}{\partial x}dx = -\rho u\frac{\partial uS}{\partial x} - \rho uS\frac{\partial u}{\partial x} \tag{1.3}$$

Ther net pressure force at the two ends is

$$-\frac{\partial(pS)}{\partial x} = -S\frac{\partial p}{\partial x} - p\frac{\partial S}{\partial x}$$

while that on the sloping wall is

$$2\pi a\, p\frac{\partial a}{\partial x} = p\frac{\partial S}{\partial x}$$

The sum of all pressure forces is

$$-S\frac{\partial p}{\partial x} \tag{1.4}$$

Balancing the momentum by equating (1.2) to the sum of a(1.3) and (1.4) we get, after making use of mass conservation (1.1),

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} \tag{1.5}$$

Let the pressure outside the artery be constant, say zero. The change in the tube radius must be caused by the change in blood pressure. Refering to figure 1, the elastic strain due to the lengthening of the circumference is $2\pi da/2\pi a = da/a$. Let h be the artery wall thickness, assumed to be must smaller than a, and Young's modulus E. The change in elastic force is $2Ehda/a$ which must be balanced by the changing in pressure force $2a\,dp$, i.e.,

$$\frac{2Eh\,da}{a} = 2a\,dp,$$

which implies

$$\frac{dp}{da} = \frac{Eh}{a^2} \quad \text{or} \quad \frac{dp}{dS} = \frac{\sqrt{\pi}Eh}{S^{3/2}} \tag{1.6}$$

Pressure increases with the tube radius, but the rate of increase is smaller for larger radius. Upon integration we get the equation of state

$$p - p_o = -E(h/a) = -\sqrt{\pi}Eh/\sqrt{S} \tag{1.7}$$

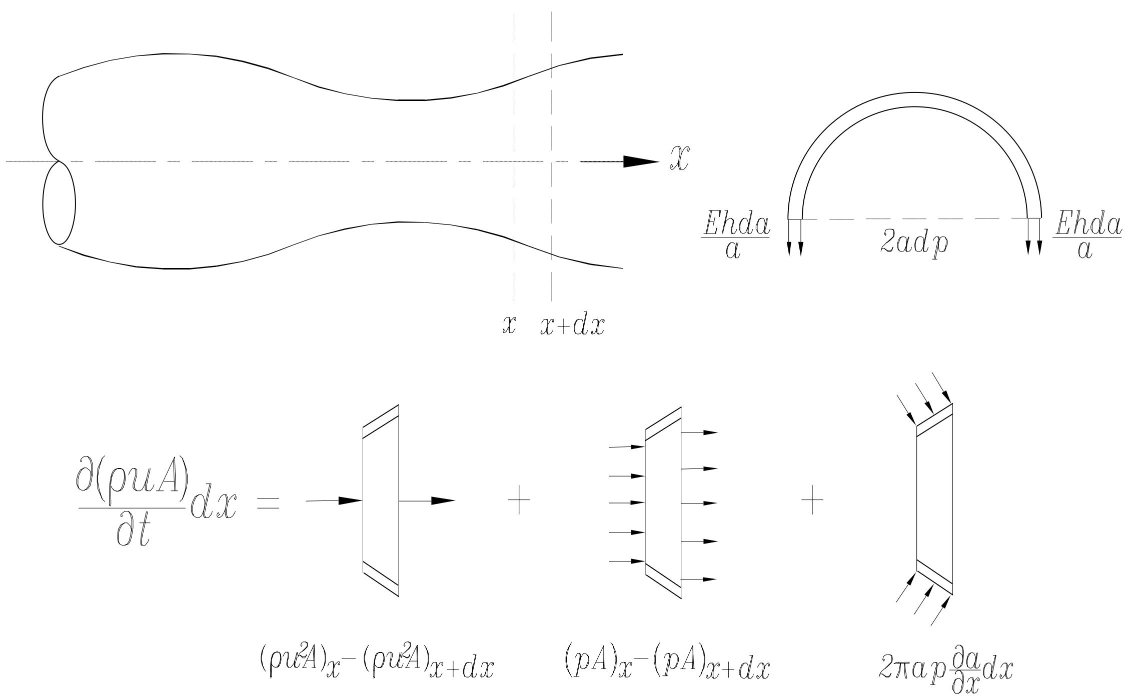

Figure 1: Forces on the artery wall.

Eq (1.5) may now be rewritten as

$$S\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{S}{\rho}\frac{\partial p}{\partial x}=-C^2\frac{\partial S}{\partial x} \tag{1.8}$$

where C is defined by

$$C=\sqrt{\frac{S}{\rho}\frac{dp}{dS}}=\sqrt{\frac{Eh}{2\rho a}} \tag{1.9}$$

and has the dimension of velocity. In view of (1.6), equations (1.1) and (1.8) are a pair of nonlinear equations for the two unknowns u and S.

For infinitesimal amplitudes we can linearize these equations. Let $a = a_o + a'$ with $a' \ll a_o$ then the (1.1) becomes, to the leading order,

$$\frac{\partial a'}{\partial t}+\frac{a_o}{2}\frac{\partial u}{\partial x}=0 \tag{1.10}$$

The linearized momentum equation is

$$\rho_o\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x} \tag{1.11}$$

The linearized form of (1.6) is

$$dp = \frac{2Eh}{a_o^2} da' \tag{1.12}$$

which can be used in (1.11) to get

$$\rho \frac{\partial u}{\partial t} = -\frac{Eh}{a_o^2} \frac{\partial a'}{\partial x} \tag{1.13}$$

Finally (1.2) and (1.8) can be combined to give the celbrated *wave equation*:

$$\frac{\partial^2 a'}{\partial t^2} = c_o^2 \frac{\partial^2 a'}{\partial x^2} \tag{1.14}$$

where

$$c_o = \sqrt{\frac{Eh}{2\rho a_o}} \tag{1.15}$$

Alternately one can eliminate a to get the equation for u

$$\frac{\partial^2 u}{\partial t^2} = c_o^2 \frac{\partial^2 u}{\partial x^2} \tag{1.16}$$

Because of (1.12), the dynamic pressure is governed also by

$$\frac{\partial^2 p}{\partial t^2} = c_o^2 \frac{\partial^2 p}{\partial x^2} \tag{1.17}$$

All unknowns are governed by the same equation due to linearity and the fact that all coefficients are constants.

To complete the formulation, initial and bounday conditions must be added. These matters will be discussed later in chapter 2.

As the process of linearization is frequently appplied in these notes, some comments are in order here. To find out the accuracy of linearization, it is useful to estimate first the scales of motion. Let A, T, L, U and P denote the scales of a', t, x, u and p' respectively. It is natural to take $L = c_o T$. From (1.1) (1.5) and (1.6) we get the relations among the scales of dynamical quantities

$$\frac{a_o A}{T} = \frac{U a_o^2}{L}, \quad \text{hence} \quad U = \frac{A}{a_o} \frac{L}{T}$$

$$P = \frac{EhA}{a_o^2}$$

$$\frac{U}{T} = \frac{1}{\rho} \frac{P}{L} = \frac{1}{\rho} \frac{1}{L/T} \frac{EhA}{a_o^2}$$

It follows that

$$\frac{A}{a_o}\frac{L}{T} = \frac{1}{\rho}\frac{1}{L/T}\frac{EhA}{a_o^2}$$

hence,

$$\frac{L^2}{T^2} = \frac{Eh}{\rho a_o} = c_o^2$$

With these scales the ratio of a typical nonlinear term to a linear term is

$$\frac{u\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial t}} \sim \frac{U^2/L}{U/T} = \frac{U}{L/T} = \frac{A}{a_o}$$

Hence the condition for linearization is that

$$\frac{A}{a_o} \ll 1$$

i.e., the amplitude of transverse oscillation is much smaller than the typical radius.

2 Sound in fluids

The basic equations governing an inviscid and compressible fluid are as follows. Mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.1}$$

Momentum conservation:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p \tag{2.2}$$

We must add an equation of state

$$p = p(\rho, S) \tag{2.3}$$

where S denotes the entropy. When no temperature gradient is imposed externally and the gradient of the flow is not too large, one can ignore thermal diffusion. The fluid motion is then adiabatic; entropy is constant. As a result $p = p(\rho, S_o)$ depends only on the density. Eq. (2.1) can be written as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\left(\frac{\partial p}{\partial \rho}\right)_S \nabla\rho \tag{2.4}$$

We shall denote

$$C = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} \tag{2.5}$$

so that

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -C^2 \nabla \rho \tag{2.6}$$

It is easy to check that C has the dimension of velocity.

From thermodynamics we also have

$$\left(\frac{\partial p}{\partial \rho} \right)_S = \gamma \left(\frac{\partial p}{\partial \rho} \right)_T \tag{2.7}$$

where T is the temperature and $\gamma = c_p/c_v$ = ratio of specific heats.

For a perfect gas the equation of state is

$$p = \rho R T \tag{2.8}$$

where R is the gas constant. Hence for a perfect gas

$$\left(\frac{\partial p}{\partial \rho} \right)_S = \gamma R T \tag{2.9}$$

Liquids are much less compressible. One usually writes the equation of state as

$$d\rho = \left(\frac{\partial \rho}{\partial p} \right)_T dp + \left(\frac{\partial \rho}{\partial T} \right)_p dT \tag{2.10}$$

Denoting

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p \tag{2.11}$$

as the coefficient of thermal expansion and

$$\kappa = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_T \tag{2.12}$$

as the coefficient of isothermal compressibility. Usually β is small and κ much smaller. Under isothermal conditions it is κ that counts.

The simplest limit is the case where the background density ρ_o and pressure ∂_o are uniform, the fluid is at rest and the dynamic perturbations are infinitesimally small. We can write

$$p = p_o + p', \quad \rho = \rho_o + \rho' \tag{2.13}$$

with $\rho' \ll \rho_o$ and $p' \ll p_o$, and linearize the equations to

$$\frac{\partial \rho'}{\partial t} + \rho_o \nabla \cdot \mathbf{u} = 0 \tag{2.14}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_o}\nabla p' \tag{2.15}$$

Taking the curl of the second, we get

$$\frac{\partial}{\partial t}\nabla \times \mathbf{u} = 0 \tag{2.16}$$

thus the velocity field is irrotational if it is so initially. We can introduce a potential ϕ by

$$\mathbf{u} = \nabla\phi \tag{2.17}$$

It follows from the momentum equation

$$p' = -\rho_o \frac{\partial \phi}{\partial t} \tag{2.18}$$

Using these we get the three-dimensional wave equation.

$$\frac{\partial^2 \phi}{\partial t^2} = c_o^2 \nabla^2 \phi \tag{2.19}$$

where

$$nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}$$

and

$$c_o = \left(\frac{\partial p_o}{\partial \rho_o}\right)_S^{1/2} \tag{2.20}$$

has the dimension of speed.

3 Shallow water waves and linearization

3.1 Nonlinear governing equations

If water in a lake or along the sea coast is disturbed, waves can be created on the surface, due to the restoring force of gravity. Consider the basic laws governing the motion of long waves in shallow water of constant density and negligible viscosity. Referring to Figure 2, let the z axis be directed vertically upward and the x, y plane lie in the initially calm water surface, $h(x, y)$ denote the depth below the still sea level, and $\zeta(x, y, t)$ the vertical displacement of the free surface. Take the differential approach again and consider the fluid flow through a vertical column with the base $dxdy$.

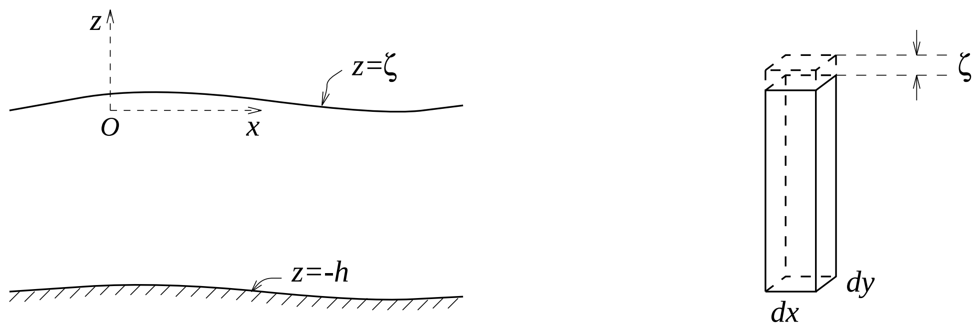

Figure 2: A column element of fluid in a shallow sea

First, the law of mass conservation. The rate of volume increase in the column

$$\frac{\partial \zeta}{\partial t} dxdy$$

must be balanced by the net volume flux into the column from all four vertical sides. In shallow water, the horizontal length scale, characterized by the wavelength λ, is much greater that the vertical length h. Water flows mainly in the horizontal planes with the velocity $\mathbf{u}(x, y, t)$, which is essentially constant in depth. Through the vertical sides normal to the x axis, the difference between influx through the left and outflux through the right is

$$-\left[u\left(\zeta+h\right)|_{x+dx} - u\left(\zeta+h\right)|_{x}\right] dy = -\left\{\frac{\partial}{\partial x}\left[u\left(\zeta+h\right)\right] + O(dx)\right\} dxdy.$$

Similarly, through the vertical sides normal to the y axis, the difference between influx through the front and outflux through the back is

$$-\left[v\left(\zeta+h\right)|_{y+dy} - v\left(\zeta+h\right)|_{y}\right] dx = -\left\{\frac{\partial}{\partial y}\left[v\left(\zeta+h\right)\right] + O(dy)\right\} dydx.$$

Omitting terms of higher order in dx, dy, we invoke mass conservation to get

$$\frac{\partial \zeta}{\partial t} dxdy = -\left\{\frac{\partial}{\partial x}\left[u\left(\zeta+h\right)\right] + \frac{\partial}{\partial y}\left[v\left(\zeta+h\right)\right] + O(dx, dy)\right\} dxdy.$$

In the limit of vanishing dx, dy, we have, in vector form,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left[\mathbf{u}(\zeta + h)\right] = 0. \tag{3.1}$$

This equation is nonlinear because of the quadratic product of the unknowns $\mathbf{u}$ and ζ.

Now the law of conservation of momentum. In shallow water the vertical momentum balance is dominated by pressure gradient and gravity, which means that the distribution of pressure is hydrostatic:

$$p = \rho g \left(\zeta - z\right) , \tag{3.2}$$

where the atmospheric pressure on the free surface is ignored. Consider now momentum balance in the x direction. The net pressure force on two vertical sides normal to the x direction is

$$\begin{aligned} dxdy\frac{\partial}{\partial x}\int_{-h}^{\zeta} p\,dz &= -dxdy\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho g(\zeta - z)\,dz \\ &= -\rho g(\zeta + h)\frac{\partial(\zeta + h)}{\partial x}dxdy. \end{aligned}$$

The hydrodynamic reaction from the sloping bottom to the fluid is

$$-p\frac{\partial h}{\partial x}dxdy = \rho g(\zeta + h)\frac{\partial h}{\partial x}dxdy.$$

The change of fluid momentum consists of two parts. One part is due to the time rate of momentum change in the water column

$$\left\{\frac{\partial}{\partial t}\left[\rho u(\zeta + h)\right]\right\} dxdy,$$

and the other is due to the net flux of momentum through four vertical sides:

$$\frac{\partial}{\partial x}[\rho u^2(\zeta + h)]dxdy + \frac{\partial}{\partial y}[\rho uv(\zeta + h)]dydx.$$

Equating the total rate of momentum change to the net pressure force on the sides and on the bottom, we get

$$\begin{aligned} &\frac{\partial}{\partial t}[\rho u(\zeta + h)] + \frac{\partial}{\partial x}[\rho u^2(\zeta + h)] + \frac{\partial}{\partial y}[\rho uv(\zeta + h)] \\ &\qquad = -g(\zeta + h)\frac{\partial(\zeta + h)}{\partial x} + g(\zeta + h)\frac{\partial h}{\partial x}. \end{aligned}$$

The left-hand side can be simplified to

$$\begin{aligned} \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)(\zeta + h) \quad &+ \quad u\left\{\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}[u(\zeta + h)] + \frac{\partial}{\partial y}[v(\zeta + h)]\right\} \\ &= \frac{\partial \zeta}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \end{aligned}$$

by invoking continuity (1.6.1). Hence the x momentum equation reduces to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \zeta}{\partial x}. \tag{3.3}$$

Similarly, momentum balance in the y direction requires

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \zeta}{\partial y}. \tag{3.4}$$

These two equations can be summarized in the vector form:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -g\nabla\zeta. \tag{3.5}$$

where ∇ is the horizontal (two-dimensional) gradient operator,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

Equations (3.1) and (3.5) are coupled nonlinear partial differential equations for three scalar unknowns $\mathbf{u}$ and ζ.

Now the boundary and initial conditions. On a shoreline S, there can be no normal flux, therefore,

$$h\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on} \quad S, \tag{3.6}$$

where $\mathbf{n}$ denotes the unit normal vector pointing horizontally into the shore. This condition is applicable not only along a cliff shore where h is finite, but also on a shoreline where $h = 0$, as long as the waves are gentle enough not to break. In the latter case the whereabout of the shoreline is unknown *a priori* and must be found as a part of the solution.

At the initial instant, one may assume that the displacement $\zeta(x, y, 0)$ and the vertical velocity of the entire free surface $\frac{\partial}{\partial t}\zeta(x, y, 0)$ is known. These conditions complete the formulation of the nonlinear shallow water wave problem.

3.2 Linearization for small amplitude

For small amplitude waves

$$\frac{\zeta}{h} \sim \frac{A}{h} \ll 1, \tag{3.7}$$

where A is the characteristic amplitude. Equation (1.6.1) may be simplified by neglecting the quadratic term

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot h\mathbf{u} = 0. \tag{3.8}$$

Denoting the time scale by the wave period T and the horizontal length scale by the wavelength λ, we equate the order of magnitudes of the remaining two terms above to get

$$\frac{A}{T} \sim \frac{uh}{\lambda}, \quad \text{implying} \quad \frac{A}{h} \sim \frac{uT}{\lambda} \ll 1.$$

Now let us estimate the importance of the quadratic term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the momentum equation by assessing the ratio

$$\frac{\mathbf{u} \cdot \nabla \mathbf{u}}{\frac{\partial \mathbf{u}}{\partial t}} = O\left(\frac{uT}{\lambda}\right) \ll 1.$$

Clearly the quadratic term representing convective inertia can also be ignored in the first approximation, and the momentum equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} = -g\nabla\zeta. \tag{3.9}$$

Both the continuity (1.6.8) and momentum (1.6.9) equations are now *linearized.*

In view of (1.6.9) the boundary condition on the shoreline (1.6.6) can be expressed, instead, as

$$h\frac{\partial \zeta}{\partial n} = 0 \quad \text{on} \quad S. \tag{3.10}$$

Consistent with the linearized approximation, the shoreline position can be prescribed *a priori.*

Equations (3.8) and (3.9) can be combined by the process of cross differentiation. First differentiate (3.8) with respect to t,

$$\frac{\partial}{\partial t}\left\{\frac{\partial \zeta}{\partial t} + \nabla \cdot (\mathbf{u}h)\right\} = 0,$$

then take the divergence of the product of (3.9) and h,

$$\nabla \cdot \left\{h\frac{\partial \mathbf{u}}{\partial t}\right\} = -\nabla(gh\nabla\zeta).$$

The difference of these two equations gives

$$\frac{\partial^2 \zeta}{\partial t^2} = \nabla \cdot (gh\nabla\zeta). \tag{3.11}$$

For a horizontal bottom $h = \text{constant}$,

$$\frac{1}{c^2}\frac{\partial^2 \zeta}{\partial t^2} = \nabla^2 \zeta, \tag{3.12}$$

where $c = \sqrt{gh} = O(\lambda/T)$ is the characteristic velocity of infinitesimal wave motion and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Equation (1.6.12) is the two-dimensional extension of the wave equation. If, furthermore, all conditions are uniform in the y direction, $\partial/\partial y = 0$, (3.12) reduces to the familiar form

$$\frac{1}{c^2}\frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 \zeta}{\partial x^2}. \tag{3.13}$$

4 Capillary-gravity waves on the sea surface

If the sea depth is not very small compared to the typical length of water waves, vertical variations in both vertical and horizontal directions can be equally important. The hydrostatic approximation in the last section is no long adequate. For very short waves, surface tension can be influential as a new restoring force. A better theory is needed.

Recall in §1 that when compressibility is included the velcotiy potential defined by $\mathbf{u} = \nabla \Phi$ is governed by the wave equation:

$$\nabla^2 \Phi = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} \tag{4.1}$$

where $c = \sqrt{dp/d\rho}$ is the speed of sound. Consider the ratio

$$\frac{\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2}}{\nabla^2 \Phi} \sim \frac{\omega^2/k^2}{c^2}$$

As will be shown later, the phase speed of the fastest surface gravity wave is $\omega/k = \sqrt{gh}$ where g is the gravitational acceleration and h the sea depth. Now h is at most 4000 m in the ocean, and the sound speed in water is $c = 1400$ m/sec^2, so that the ratio is at most

$$\frac{40000}{1400^2} = \frac{1}{49} \ll 1$$

As a good approximation we replace (4.1) by

$$\nabla^2 \Phi = 0 \tag{4.2}$$

which amounts to ignoring compressibility.

The above result can of course also be derived from the basic conservation laws for incompressible fluids. The law of mass conservation then reads

$$\nabla \cdot \mathbf{u} = 0 \tag{4.3}$$

For small amplitude motion, the linearized momentum equation reads

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla P - \rho g \mathbf{e}_z \tag{4.4}$$

Now let the total pressure be split into static and dynamic parts

$$P = p_o + p \tag{4.5}$$

where ∂_o refers to the static pressure

$$p_o = -\rho g z \tag{4.6}$$

satisfying

$$0 = -\nabla p_o + -\rho g \mathbf{e}_z \tag{4.7}$$

It follows that

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p \tag{4.8}$$

Clearly the velocity field is irrotational if it is so initially, hence $\mathbf{u} = \nabla \Phi$ and (4.2) follows from (4.3). Note that

$$p = -\rho \frac{\partial \Phi}{\partial t} \tag{4.9}$$

Refering to Figure **??**. let the free surface be $z = \zeta(x, y, t)$. Then for a gently sloping free surface the vertical velocity of the fluid on the free surface must be equal to the vertical velocity of the surface itseld. i.e.,

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z}, \quad z = 0. \tag{4.10}$$

Having to do with the velocity only, this is called the *kinematic boundary condition.*

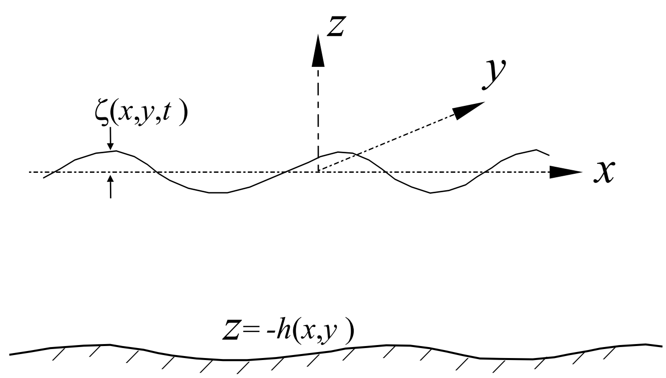

Figure 3: Definition sketch for surface wave problem

Let us assume that the air above the sea surface is essentially stagnant, because of its relative small density we ignore its presence and assume the air pressure to be zero. If surfrace tension is ignored continutiy of pressure requires that

$$p = p_o + p = 0, \quad z = \zeta.$$

to the leading order of aproximation, we have

$$\rho g \zeta + \rho \frac{\partial \Phi}{\partial t} = 0, \quad z = 0. \tag{4.11}$$

Being a statement on forces, this is called the *dynamic boundary condition*.

The two conditions (4.10) and (4.11) can be combined to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0, \quad z = 0 \tag{4.12}$$

If surface tension is also included then we adopt the model where there is a thin film covering the wter surface with tension T per unit length. Consider a unit square $dxdy$ on the free surface. the net vertical force from four sides is

$$\left(T \frac{\partial \zeta}{\partial x}\bigg|_{x+dx} - T\, \frac{\partial \zeta}{\partial x}\bigg|_{x} \right) dy + \left(T\, \frac{\partial \zeta}{\partial y}\bigg|_{y+dy} - T\, \frac{\partial \zeta}{\partial x}\bigg|_{y} \right) dx = T \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) dx\, dy$$

Conitinuity of vertical force on an unit area of the surface requires

$$p_o + p + T \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) = 0.$$

Hence

$$-\rho g \zeta - \rho \frac{\partial \Phi}{\partial t} + T \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) = 0, \quad z = 0. \tag{4.13}$$

which can be combined with the kinematic condition (4.10) to give

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} - \frac{T}{\rho}\frac{\partial^3 \Phi}{\partial x^2 \partial z} = 0, \quad z = 0 \tag{4.14}$$

When viscosity is neglected, the normal fluid velocity vanishes on the rigid seabed,

$$\mathbf{n} \cdot \nabla \Phi = 0 \tag{4.15}$$

Let the sea bed be $z = -h(x, y)$ then the unit normal is

$$\mathbf{n} = \frac{(h_x, .h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}} \tag{4.16}$$

Hence

$$\frac{\partial \Phi}{\partial z} = -\frac{\partial h}{\partial x}\frac{\partial \Phi}{\partial x} - \frac{\partial h}{\partial y}\frac{\partial \Phi}{\partial y}, \qquad z = -h(x, y) \tag{4.17}$$

Homework No.1

1. During an earthquake, water in a reservoir exerts hydrodynamic pressure on a dam that may fail. Formulate the dam-reservoir interaction problem under the following idealizations. The reservoir is infinitely long and has a uniform rectangular cross section. Water is present only on one side of the dam ($x > 0$) and has the constant depth h. Before $t = 0$, all is calm. After $t = 0$ the dam is forced to vibrate horizontally so that

$$u(0, y, z, t) = \begin{cases} u_o(y, z, t) = \text{prescribed}, & 0 < t < T, \\ 0, & t > T. \end{cases} \tag{4.1}$$

The free surface is exposed to constant atmospheric pressure. The reservoir bottom is rigid and does not vibrate vertically (!!!). Neglect gravity but consider compressibility of water because of the high frequency ($\sim O(100)$Hz). Express all governing equations including the boundary conditons in terms of the velocity potential ϕ defined by $(u, v, w) = \nabla \phi$.