On the Effectiveness of Dark Energy and General Relativity in Modeling the Universe

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Abstract

The cause of the accelerating expansion of the universe has drawn much speculation in recent years. Dark energy is the most popular explanation for this phenomenon. Using data from Type Ia Supernovae, models of a universe with and without dark energy are contrasted to determine which explanation is more accurate. While universes with dark energy model the data more closely than ones without, the possibility for viable alternatives through modifications to general relativity remains strong.

1 Introduction

One of the more intriguing behaviors of the universe is that its expansion is presently accelerating contrary to Newtonian physics and the Newtonian gravity [3]. One possible explanation for this is that a mysterious "dark energy" which drives the galaxies apart [8]. However, dark energy may not be the only possible explanation.

The basis of the theory of dark energy begins with redshift. The expansion of the universe is detected through the distortion of the light emitted by the receding galaxies [3]. Due to Doppler shifts, wavelengths received from galaxies moving away from the Milky Way are elongated and contain less energy. The observed redshift indicates that most galaxies are moving away from the Milky Way. Comparison of galaxies' luminosity and apparent brightness to those of standard candles (galaxies of known luminosity and class) allows redshift to be related to the distance from the galaxy to Earth. In 1929, Hubble used these relationships to discover that a galaxy's recession velocity increase with its distance from Earth, thus establishing the expansion of the universe [4]. Nearly seventy years passed, though, until in 1998 astronomers were able to relate time, distance and velocity to determine that the expansion accelerates with time [8]. Not impossible within the context of General Relativity, this recent relevation has created a flurry of new theories in attempts to model our expanding and accelerating universe.

In a Newtonian world, matter attracts matter as related by Newton's Gravitational Law. When gravitational acceleration is applied to the scale of the universe, the Gravitational Law becomes:

$$\frac{d^2a}{dt^2} = -\frac{4G\pi}{3}\rho a \ , \tag{1}$$

where G is the gravitational constant, ρ is the density of the universe and a is the an arbitrary and dimensionless measure of the expansion rate of the universe [8]. Assuming that the density of the universe ρ is positive, all possible predictions for the acceleration of expansion of the universe d^2a/dt^2 are negative, indicating the universe should be decelerating. Contrary to Newton, cosmologists have established that the universe is instead accelerating. For this reason among others, Einstein's theory of General Relativity is accepted as the correct theory of gravity in the universe.

The general relativistic equation of gravity is expressed as

$$\frac{d^2a}{dt^2} = -\frac{4}{3}\pi Ga\left(\rho + \frac{3P}{c^2}\right), \tag{2}$$

where P represents the total pressure present in the universe[8]. Published in 1916, this equation incorporated pressure as a primary determinant of acceleration. Should the pressure of the universe be positive, the acceleration would be negative, predicting deceleration of the universe and eventually recolapse. However, when cosmologists observed acceleration instead, they could only conclude that an undiscovered source of massive negative pressure, a source so large as to overwhelm the attraction of matter, existed [7]. These sources of negative pressure are called dark energy.

The idea of dark energy, defined as any substance with negative pressure, was created by cosmologists to explain acceleration. As a substance that neither interacts with light nor affects objects on small scales yet is present homogeneously throughout space, dark energy is thought to contribute at least two-thirds of the mass-energy density of the universe [2]. Its tremendous negative pressure would cause it to repel galaxies rather than attract them. However, definitive observation of dark energy has proved elusive thus far. That the only evidence for the existence of dark energy comes from observational cosmology has contributed to speculation that dark energy may not be neither the best nor correct explanation [8]. Alternative views and models may provide a better model for the universe holistically.

This paper uses the data of standard candles to assess the accuracy of dark energy as the source of repulsive gravity. A modification of general relativity will also be proposed and assessed to test the applicability of General Relativity at large length scales. From these tests, our models of the universe may be revised and strengthened to create a more accurate depiction of the universe.

In Section 2, we first examine the theoretical basis of dark energy theories, including redshift, luminosity distance, radial distance, space curvature and time. Section 3 uses the Friedmann equation to interpret data from standard candles as well as establish a model of a universe of dark energy. The data and analysis of the dark energy model is presented in Section 4. A modified theory of general relativity is postulated in Section 5 to model a universe without dark energy. Our conclusions are presented in Section 6.

2 Theoretical Background

In order to test dark energy, a relationship between redshift, cosmic expansion, distance and time must first be established. The Friedmann equation for acceleration can then be manipulated to model data taken from standard candles to determine the effectiveness of dark energy as a model of the universe.

2.1 Redshift

The first essential concept in models of dark energy is redshift. We follow the presentation of John Hawley [3] in its discussion. Redshift itself is the stretching of light emitted from galaxies due to the emitting object's recession. Cosmic expansion, denoted by a, cause the distances to increase, so if the ratio of the emitted wavelength λ to the observed wavelength is directly related to the ratio of cosmic expansion at the time when the light was emitted (t_{em}) to when it was observed (t_{obs}), then

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} \ , \tag{3}$$

where z represents redshift. Define $t_{obs} = t_0$, the subscript "0" indicating today's time after the Big Bang, at which $t = 0$ and consider $a(t_0) = 1$, then cosmic expansion can be expressed as a function of redshift:

$$1 + z = \frac{1}{a(t_{em})} \ . \tag{4}$$

2.2 Space Curvature k and Radial Distance χ

The second important concept is distance and shape in spacetime. Distance in the universe depends upon the curvature k of space. In a flat space $k = 0$, in a hyperbolic space $k < 0$, and in a spherical space $k > 0$. Current measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) favor a k value of zero [2]. We will assume this to be true. A proper metric for distance can now be derived.

Assuming flat spacetime with no gravity, the Minkowski metric for distance ds in four dimensions (x, y, z, and t) is given by

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{5}$$

In a static universe, this form of the metric would adequately describe the distance between stars. However, the universe is expanding. As in a balloon, the coordinates x, y, z in the universe remain constant; however, as more "air" is added to the balloon, the distance increases proportional to the expansion factor a. The metric becomes

$$ds^2 = -c^2 dt^2 + a^2(t) \left[dx^2 + dy^2 + dz^2 \right] \ . \tag{6}$$

In spherical coordinates, this becomes

$$ds^2 = -c^2 dt^2 + a^2(t) \left[d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2) \right] \ , \tag{7}$$

where $r(\chi)$ is the radial distance to a point whose polar and azimuthal angles are given by θ and ϕ, respectively. This metric is known as the Robertson-Walker metric. The definition of $r(\chi)$ varies with the value of the curvature of space k [3]. In this paper, $k = 0$; for this value of k, the term $r(\chi)$ simply reduces to χ in the metric.

For the radial path of a photon moving toward the origin along a light ray, the metric reduces to

$$ds^2 = 0 = -c^2 dt^2 + a^2(t) d\chi^2 \ . \tag{8}$$

As the speed of light is c, then along the light ray, $d\chi = -cdt/a(t)$. By integrating this speed,

$$\int_0^{\chi_{em}} d\chi = -c \int_{t_0}^{t_{em}} \frac{dt}{a(t)} \ , \tag{9}$$

the distance χ can be written as a function of absolute time t, measured since the beginning of the universe and time.

2.3 Luminosity Distance d_L

The luminosity of a galaxy, a measure proportional to the radial distance, is normally the only way to experimentally measure distance. This is important for interpreting the data given by the standard candles. Flux is inversely proportional to the distance the light traveled squared and directly related to the luminosity of the galaxy. So, the flux is

$$S = \frac{L}{4\pi d_L^2} \ , \tag{10}$$

where d_L is the luminosity distance and L is the luminosity of the object. Cosmological redshift reduces the energy of the emitted photons by a factor of a, or $1/(1+z)$, as expansion increases the distance the photons travel. Furthermore, as the emitting source moves away from the point of observation, the frequency of the photons is also decreased by a factor of

a. Therefore, adjusted for cosmological redshift,

$$S = \frac{L}{4\pi r^2(\chi_{em})(1+z)^2} \ , \tag{11}$$

where $d_L = (1+z) \cdot r(\chi_{em}) = (1+z) \cdot \chi_{em}$.

2.4 Absolute and Conformal Time

Time is essential to the discussion of acceleration. However, determining the rate of accelerating a as a function of t_{em}, as written in equation (9) is not feasible with the current knowledge and measurements of the universe. Astronomers cannot measure the time when objects emitted their light; they can only measure redshift and luminosity distance. An indirect method known as conformal time, a measure of time relative to the age of the universe instead of absolute t since the Big Bang, allows us to remove the barrier presented by unknown absolute times. Once conformal time is substituted for the absolute measure, we will be able to model the acceleration of the universe and test dark energy.

Define conformal time τ as $\tau(t) = \int_0^t a^{-1} dt$. Utilization of this definition in equation (11),transforms it into

$$\chi_{em} = c(\tau_0 - \tau_{em}) \ . \tag{12}$$

Now, through use of conformal time, redshift, and the Robertson-Walker metric, the radial distance χ can be expressed as a function of time, allowing models of the accelerating universe to be formed.

3 The Friedmann Equation and Dark Energy

In general relativity, the acceleration of the universe follows the Friedmann equation. It can be derived from both Newtonian physics and general relativity. For the universe, the general relativistic form of the Friedmann equation is given by [8]

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} \ . \tag{13}$$

As neither τ nor t can be measured by astronomers, if we can use redshift and changes in the scale factor a as relative measures of time, thereby replacing τ, then we can test the equation and thereby test general relativity and dark energy as a model of the accelerating universe.

Written in terms of conformal time, equation (13) becomes

$$\left(\frac{1}{a^2}\frac{da}{d\tau}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} \ . \tag{14}$$

The derivative of equation (14), taken with respect to conformal time, is equal to equation (2). Solving equation (14) for τ yields the equation

$$\tau = \int_0^a \frac{da}{\sqrt{\frac{8\pi}{3}G\rho a^4 - ka^2}}. \tag{15}$$

The total density of the universe is $\rho = \rho_M + \rho_\Lambda$. The amount of matter in the universe is a constant. As the volume of space in the universe increases, the mean density of matter (denoted by subscript "M") decreases proportional to expansion, so $\rho \equiv \rho_{M,0}a^{-3}$. The density of the dark energy, though, is a constant [8]. The symbol Λ has been introduced to represent dark energy; it is also known as the cosmological constant or vacuum energy.

The ratios of matter and dark energy in the universe to the critical density, which determines curvature, are respectively given by

$$\Omega_M = \frac{8\pi}{3}\frac{G\rho_{M,0}}{H_0^2} \ , \qquad \Omega_\Lambda = \frac{8\pi}{3}\frac{G\rho_\Lambda}{H_0^2} \ . \tag{16}$$

Assuming space is flat, $\Omega_M + \Omega_\Lambda = 1$. Then, if $\frac{8\pi}{3} G\rho a^4 = H_0^2\left(\Omega_M a + \Omega_\Lambda a^4\right)$, where the Hubble Constant $H_0 = 71$ km s^{-1} Mpc^{-1} [3], substitution of χ for τ results in

$$\chi = \frac{c}{H_0} \int_a^1 \frac{da}{\sqrt{\Omega_M a + \Omega_\Lambda a^4}} \ . \tag{17}$$

Equation (17), depending upon the amount of matter and dark matter in the universe, expresses the radial distance to an object. The goal of the dark energy model in this paper is to compute the values of Ω_M and Ω_Λ which most accurately model the data from Type Ia Supernovae.

4 Supernovae Data

The redshift and luminosity distance of 231 Type Ia Supernovae, collected and published in April, 2003 [7], plot $H_0\chi$ versus a.

Two of the possible models resulting from the integral in equation (17) for χ, one with $\Omega_M = 1$ and the other with $\Omega_M = 0.27$ (the value favored by the WMAP observations [2]), are also plotted on Figure 1. The supernovae data, binned with respect to $\Delta a = .05$ beginning at $a = 0$, are plotted together with their error bars.

To find the optimal value of Ω_M in equation (17), a χ^2 statistical test for goodness of fit was performed. The equation for a standard χ^2 test is

$$\chi^2 = \sum \frac{\left(x_{\text{obs}} - x_{\text{predicted}}\right)^2}{\text{variance}} \ . \tag{18}$$

However, for the supernova data, a correction to the variance must be made to account for peculiar motion. Peculiar motion is the gravity-induced velocity of a galaxy due to the attraction of matter. The velocity, included in the measurements, changes the redshift of a supernova from the value it would have in a homogeneous and isotropic expanding universe. By adding 500 km s^{-1} to the variance, typical of peculiar motions [9], the variance will account for this extra motion in assessing the fit of the model. Also, as the original measurements instead of the ones from the equations are thought to have an approximately Gaussian distribution, the χ^2 test shall be performed using the observed $\log(H_0 d_L)$ values. After adjusting for peculiar motion and the transformation to logs, the test becomes

$$\chi^2 = \sum \frac{\left[\log(H_0 d_L)^{obs} - \log\left(\frac{H_0\chi}{a}\right)\right]^2}{\text{variance} + \left(\frac{500}{c\cdot z\cdot ln(10)}\right)^2}\ , \tag{19}$$

where the variance refers to $\log_{10}(H_0 d_L)$ and $\chi(a)$ is computed from equation (17).

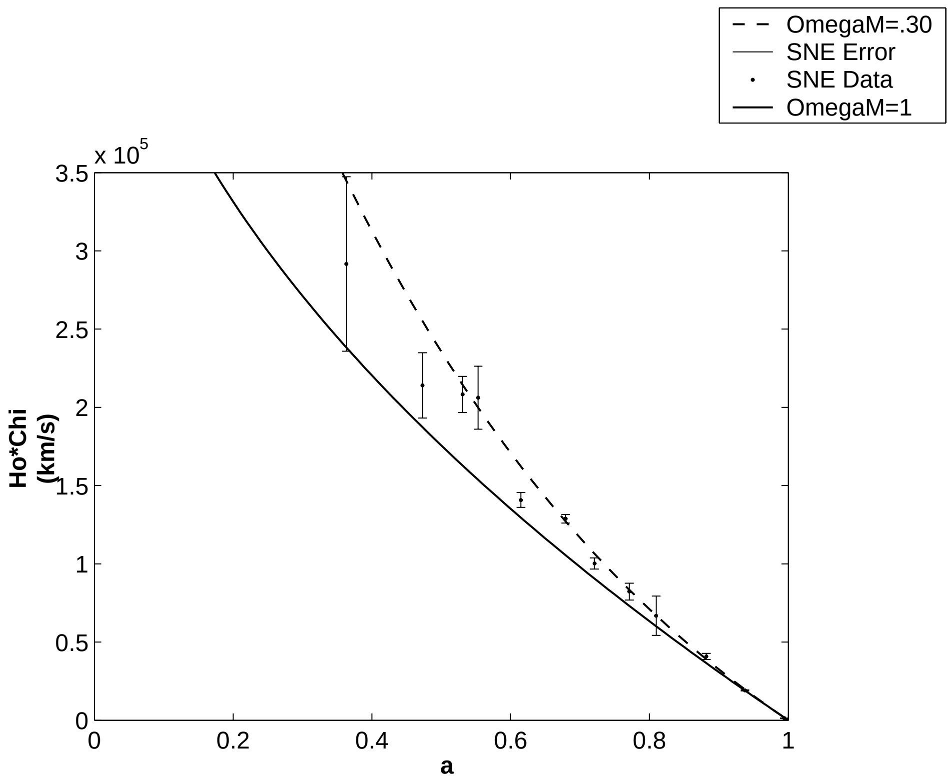

Figure 1: The binned supernovae data plot observed $H_0\chi$ versus a alongside the theoretical models, one with $\Omega_M = .27$ and the other with $\Omega_M = 1$.

Supernova which had a redshift of $z < .01$ were excluded to eliminate systematic errors.

At such small redshift, the peculiar motion is too great in comparison to the corresponding low recession velocity. Also, the reddening value indicates how much dimming of the supernova's emitted light is caused by dust, which artificially increases the estimated distance. To prevent inclusion of data overly affected by reddening, supernovae with reddening values $A_V > 0.5$ were also excluded. After the exclusion of these supernovae, there are $n = 172$ supernovae left to test.

When the χ^2 test statistic is plotted against Ω_M, Figure 2 results. The χ^2 value should

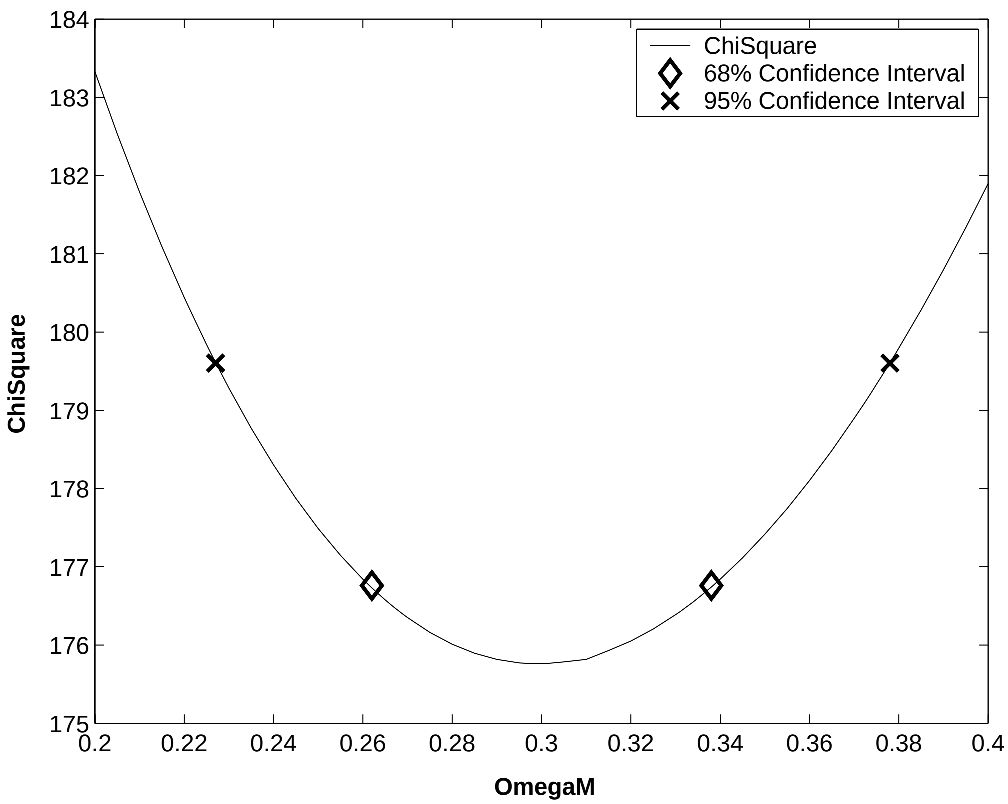

Figure 2: The χ^2 curve. The minimum occurs at $\Omega_M = .30$. The 68% confidence level is $+/-.04$ and the 95% confidence level is $+/-.08$.

equal the number of degrees of freedom $(n-1)$ if the model is correct. Each data point should ideally have an average χ^2 contribution of one, which indicates the model predicts a value which is one standard deviation away from the observed. This measure of differences between observed and predicted are how χ^2 show goodness of fit. Should the model be incorrect, though, the value of χ^2 increases. The minimum of χ^2 also indicates model which gives values the closest to the observed data; therefore, where χ^2 is smallest, that value of Ω_M is optimal. In Figure 2, the minimum of χ^2 occurs at $\Omega_M = .30$ where $\chi^2 = 175.76$. This minimum is also consistent with the expected value for χ^2, indicating the model is a good fit overall.

The uncertainty in the parameters obtained from the χ^2 uses the maximum likelihood method[1]. At a 68% confidence level, the uncertainty in Ω_M is 0.04. The uncertainty is 0.08 at the 95% confidence level.

The value of Ω_M favored by the WMAP observations is 0.27, with an uncertainty of 0.04 [2]. The minimum of χ^2 obtained from our model is within one standard deviation of the WMAP values, which indicates that the model is reasonable, as 34% of the WMAP predictions would lie within one standard deviation above its value of 0.27. Yet another calculation released in March of 2003 estimated that $\Omega_M = 0.28$, with an error bar of 0.05 [9]. However, both this paper and the Tonry paper used the same data and exclusions in calculating Ω_M, yet arrived at different conclusions. The difference could have arisen if the Tonry paper calculated χ^2 using $H_0 d_L$ instead of $\log(H_0 d_L)$.

The value $\Omega_M = .30$ means that 70% of the mass-energy in the universe is in the form of dark energy. If dark energy accounts for that much of the mass-energy of the universe, it is difficult to accept that dark energy has not been physically observed yet. Indeed, dark energy may not be the correct theory. In the next section of this paper, we consider an alternative model of the universe in which dark energy does not exist.

5 Another Theory of Gravity

In the months following Einstein's publication of general relativity, a mathematician named Hilbert formulated a concise derivation of the theory; it was called the least-action principle of gravity. The least-action principle requires one to minimize $\int \mathcal{L}\sqrt{-g}\, d^4x$, with the Lagrangian $\mathcal{L} = R \cdot 16\pi G$ where R is the Ricci scalar, defined later in this section.

The modifications to the General Theory of Relativity here are postulated in a natural way by replacing R in the Lagragian by an abitrary function $f(R)$. In doing so, the original theory can be recovered in the event that there is no feasible alternative model by setting $f(R) = R$. The changes yield the following two equations:

$$-3\left(\frac{df}{dR}\right)\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) + \frac{1}{2}a^2 f = 8\pi G a^2 \rho \ , \tag{20}$$

$$\frac{df}{dR}\left[\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) + 2\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 + 2k\right] - \frac{1}{2}a^2 f = 8\pi G a^2 \frac{P}{c^2} \ . \tag{21}$$

The Ricci scalar for the Robertson-Walker metric is given by

$$R \equiv \frac{6}{a^2}\left[\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) + \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 + k\right] \ . \tag{22}$$

In these modifications, $f(R)$ is function which has units of $[H_0^2]$, or equivalently, $1/\tau^2$. It is arbitrary because we don't know of any certain alternatives to dark energy in the universe. We assume that $P = 0$ as dark energy is the only reason why $P < 0$ in the universe. We also assume $\rho = \rho_M$ as $\rho_\Lambda = 0$ if there is no dark energy in the universe. The function $f(R)$ must, though, have the property $\lim_{f \to \infty} = R$ in order to recover Newtonian gravity. Two possible models for $f(R)$ which will be used in this paper are

$$f = R + R_0 \quad , \tag{23}$$

and

$$f = R + \frac{R_1}{R} \ , \tag{24}$$

where R_0 and R_1 are constants.

Next we consider the consequences and potential of each of these models to see how feasible a universe without dark energy is.

5.1 Alternative One

Let us first consider the equation (23). Substitution into equations (20) and (21) transform those equations into

$$-3\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) + \frac{1}{2}a^2\left(R_0 + R\right) = 8\pi G a^2 \rho \ , \tag{25}$$

and

$$\left[\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) + \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 + 2k\right] - \frac{1}{2}a^2\left(R_0 + R\right) = 8\pi G a^2 \frac{P}{c^2} \ . \tag{26}$$

Using the definition of the Ricci scalar, equation (25) becomes

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \frac{8\pi}{3}G^2 a^2 \rho - k - \frac{1}{6}a^2 R_0 \ . \tag{27}$$

Equation (26) becomes

$$-2\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) - \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = 8\pi G a^2 \frac{P}{c^2} + k + \frac{1}{2}a^2 R_0 \tag{28}$$

We know from observations that the acceleration of the universe today is positive. If $P = 0$, as it does in a universe without dark energy, then the only way to achieve positive acceleration in equation (27) is if $R_0 < 0$. However, this solution is not distinguishable from the Friedmann equation with dark energy.

By factoring equation (27), we get

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \frac{8\pi}{3}G^2\left(\rho - \frac{R_0}{16\pi G a^2}\right) - k\ . \tag{29}$$

If $\rho = \rho_M + \rho_\Lambda$, then in the term $\rho - R_0/(16\pi G)$, ρ_Λ corresponds to $-R_0/(16\pi G)$. If we substitute this into equation (27),

$$-2\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) - \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = 8\pi G a^2\left(\frac{P}{c^2} + -\rho_\Lambda\right) + k \tag{30}$$

The term $p/c^2 - \rho_\Lambda$ is equivalent to vacuum energy which has $P_\Lambda = -\rho_\Lambda c^2$. We conclude that R_0 is indistinguishable from dark energy. Instead, alternative two may provide a model without dark energy.

5.2 Alternative Two

Alternative two is quite different than alternative one. In this model, recall

$$f = R + \frac{R_1}{R}, \quad \text{therefore} \quad \frac{df}{dR} = -\frac{R_1}{R^2} + 1\ . \tag{31}$$

Substitution of these equations into equation (20) then gives

$$\left(\frac{-R_1}{R^2} + 1\right)\left[\frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) + 2\left(\frac{1}{a^2}\frac{da}{d\tau}\right)^2\right] = 0\ , \tag{32}$$

assuming $k = P = 0$. We let

$$\alpha = 2H_0^2 = 2\left(\frac{1}{a^2}\frac{da}{d\tau}\right)^2\ . \tag{33}$$

Then equation (21) becomes

$$R^3 - 3\alpha R^2 + 2R_1 R + 3R_1\alpha = 0\ . \tag{34}$$

The shape of a cubic formula is given by $\delta^2 = (b^2 - 3ac)(9a^2)$ [5]. For $\delta^2 > 0$, the maxima and minima are distinct; for $\delta^2 = 0$ they coincide, and for $\delta^2 < 0$ the graph contains no turning points. The point of inflection or symmetry is given by $x_n = -b/(3a)$. As α is a function dependent on time and R_1 is a constant, the sign of δ^2 changes with time. Also, as the universe was decelerating in the past, the value of H was also larger in the past than at the present. Research in the solutions and shape of equation (34) was not finished, unfortunately; however, it remains an intriguing possibility for modeling the universe.

6 Conclusion

The data collected from the Type Ia supernovae indicate that the theory of dark energy, despite its lack of physical proof, provides a very accurate model of the universe when 70% of the mass-energy density of the universe is dark energy. Although our first alternative model to dark energy was indistinguishable from dark energy, dozens of other possibilities such as the second alternative posed here remain to be explored and analyzed. Continued exploration into the nature of R_1 and other functions of $f(R)$ will reveal more about the use of dark energy and general relativity in modeling the universe.

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