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Hamiltonian Dynamics of Particle Motion

1 Introduction

These notes present a treatment of geodesic motion in general relativity based on Hamilton's principle, illustrating a beautiful mathematical point of tangency between the worlds of general relativity and classical mechanics.

2 Geodesic Motion

Our starting point is the standard variational principle for geodesics as extremal paths. Adopting the terminology of classical mechanics, we make the action stationary under small variations of the parameterized spacetime path $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$ subject to fixed values at the endpoints. The action we use is the path length:

$$S_1[x(\tau)] = \int \left[g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2} d\tau \equiv \int L_1(x, dx/d\tau)\, d\tau \ . \tag{1}$$

Variation of the trajectory leads to the usual Euler-Lagrange equations

$$\frac{d}{d\tau} \left[\frac{\partial L}{\partial(dx^\mu/d\tau)} \right] - \frac{\partial L}{\partial x^\mu} = 0 \ , \tag{2}$$

from which one obtains the equation of motion

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \frac{1}{L_1} \frac{dL_1}{d\tau} \frac{dx^\mu}{d\tau} = 0 \ . \tag{3}$$

The last term arises because the action of equation (1) is invariant under arbitrary reparameterization. If the path length is taken to be proportional to path length, $d\tau \propto$

$ds = (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$, then $L_1 = ds/d\tau =$ constant and the last term vanishes, giving the standard geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \ . \tag{4}$$

It may be shown that any solution of equation (3) can be reparameterized to give a solution of equation (4). Moreover, at the level of equation (4), we needn't worry about whether τ is an affine parameter; we will see below that for any solution of equation (4), τ is automatically proportional to path length. The full derivation of the geodesic equation and discussion of parameterization of geodesics can be found in most general relativity texts (e.g. Misner et al 1973, §13.4).

The Lagrangian of equation (1) is not unique. Any Lagrangian that yields the same equations of motion is equally valid. For example, equation (4) also follows from

$$S_2[x(\tau)] = \int \frac{1}{2} g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\, d\tau \equiv \int L_2(x, dx/d\tau)\, d\tau \ . \tag{5}$$

Unlike equation (1), which is extremal for geodesic curves regardless of their parameterization, equation (5) is extremal for geodesics only when τ is an affine parameter, $d\tau/ds =$ constant. In other words, τ measures path length up to a linear rescaling.

The freedom to linearly rescale the affine parameter allows us to define τ so that $p^\mu = dx^\mu/d\tau$ gives the 4-momentum (vector) of the particle, even for massless particles for which the proper path length vanishes. One may easily check that $d\tau = ds/m$ where m is the mass.

With the form of the action given by equation (5), the canonical momentum conjugate to x^μ equals the momentum one-form of the particle:

$$p_\mu \equiv \frac{\partial L_2}{\partial(dx^\mu/d\tau)} = g_{\mu\nu} \frac{dx^\nu}{d\tau} \ . \tag{6}$$

The coincidence of the conjugate momentum with the momentum one-form encourages us to consider the Hamiltonian approach as an alternative to the geodesic equation. In the Hamiltonian approach, coordinates and conjugate momenta are treated on an equal footing and are varied independently during the extremization of the action. The Hamiltonian is given by a Legendre transformation of the Lagrangian,

$$H(p, x, \tau) \equiv p_\mu \frac{dx^\mu}{d\tau} - L(x, dx/d\tau, \tau) \tag{7}$$

where the coordinate velocity $dx^\mu/d\tau$ must be expressed in terms of the coordinates and momenta. For Lagrangian L_2 this is simple, with the result

$$H_2(p_\mu, x^\nu, \tau) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu \ . \tag{8}$$

Notice the consistency of the spacetime tensor component notation in equations (6)-(8). The rules for placement of upper and lower indices automatically imply that the conjugate momentum must be a one-form and that the Hamiltonian is a scalar.

The reader will notice that the Hamiltonian H_2 exactly equals the Lagrangian L_2 (eq. 5) when evaluated at a given point in phase space (p, x). However, in its meaning and use the Hamiltonian is very different from the Lagrangian. In the Hamiltonian approach, we treat the position and conjugate momentum on an equal footing. By requiring the action to be stationary under independent variations $\delta x^\mu(\tau)$ and $\delta p_\nu(\tau)$, we obtain Hamilton's equations in four-dimensional covariant tensor form:

$$\frac{dx^\mu}{d\tau} = \frac{\partial H_2}{\partial p_\mu} \ , \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H_2}{\partial x^\mu} \ . \tag{9}$$

Evaluating them using equation (8) yields the canonical equations of motion,

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu} p_\nu \ , \quad \frac{dp_\mu}{d\tau} = -\frac{1}{2}\frac{\partial g^{\kappa\lambda}}{\partial x^\mu} p_\kappa p_\lambda = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} g^{\kappa\alpha} g^{\beta\lambda} p_\kappa p_\lambda = g^{\beta\lambda} \Gamma^\kappa{}_{\mu\beta} p_\kappa p_\lambda \ . \tag{10}$$

These equations may be combined to give equation (4).

The canonical equations (9) imply $dH/d\tau = \partial H/\partial\tau$. Because H_2 is independent of the parameter τ, it is therefore conserved along the trajectory. Indeed, its value follows simply from the particle mass:

$$g^{\mu\nu} p_\mu p_\nu = -m^2 \ \rightarrow \ H_2(p, x) = -\frac{1}{2} m^2 \ . \tag{11}$$

It follows that solutions of Hamilton's equations (10) satisfy $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \propto d\tau^2$, hence τ must be an affine parameter.

At this point, it is worth explaining why we did not use the original, parameterization-invariant Largrangian of equation (1) as the basis of a Hamiltonian treatment. Because L_1 is homogeneous of first degree in the coordinate velocity, $(dx^\mu/d\tau)\partial L_1/\partial(dx^\mu/d\tau) = L_1$ and the Hamiltonian vanishes identically. This is a consequence of the parameterization invariance of equation (1). The parameterization-invariance was an extra symmetry not needed for the dynamics. With a non-zero Hamiltonian, the dynamics itself (through the conserved Hamiltonian) showed that the appropriate parameter is path length.

3 Separating Time and Space

The Hamiltonian formalism developed above is elegant and manifestly covariant, i.e. the results are tensor equations and therefore hold for any coordinates and any reference frame. However, the covariant formulation is inconvenient for practical use. For one thing, every test particle has its own affine parameter; there is no global invariant clock by which to synchronize a system of particles. Sometimes this is regarded, incorrectly, as a shortcoming of relativity. In fact, relativity allows us to parameterize the spatial position of any number of particles using the coordinate time $t = x^0$. (After all, time was invented precisely to label spacetime events with a timelike coordinate.) An observer would report the results of measurement of any number of particle trajectories as $x^i(t)$; there is no ambiguity nor any loss of generality as long as we specify the metric.

Our goal is to obtain a Hamiltonian on the six-dimensional phase space $\{p_i, x^j\}$ which yields the form of Hamilton's equations familiar from undergraduate mechanics:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \ , \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \ . \tag{12}$$

However, unlike undergraduate mechanics, we require that these equations of motion be fully correct in general relativity. Their solutions must be consistent with solutions of equation (10). We might hope simply to eliminate τ as a parameter, replacing it with t, while retaining the spatial components of p_μ and x^ν for our phase space variables. But what is the Hamiltonian, and can we ensure relativistic covariance?

The answer comes from a third expression for the action, regarded now as a functional of the 6-dimensional phase space trajectory $\{p_i(t), x^j(t)\}$:

$$S_3[p_i(t), x^j(t)] = 2S_2 = \int p_\mu dx^\mu = \int \left(p_0 + p_i \frac{dx^i}{dt} \right) dt \ . \tag{13}$$

Note that S_3 is manifestly a spacetime scalar, but that we have separated time and space components of the momentum one-form. Our desire to have a global time parameter has forced this space-time split.

Equation (13) is highly suggestive if we recall the Legendre transformation $H = p_i dx^i/dt - L$ (written here for three spatial coordinates parameterized by t rather than four coordinates parameterized by τ). Inverting the transformation, we conclude that the factor in parentheses in equation (13) must be the Lagrangian so that $S_3 = \int L\, dt$, and therefore the Hamiltonian is $H = -p_0$.

This result is appealing: the Hamiltonian naturally works out to be (minus) the time component of the momentum one-form. It is suggestive that, in locally flat coordinates, $-p_0 = p^0$ is the energy. However, despite appearances, the Hamiltonian is *not* in general the proper energy. Our formalism works for arbitrary spacetime coordinates and is not restricted to flat coordinates or inertial frames. We only require that t be time-like so that it can parameterize timelike spacetime trajectories.

Equation (13) with $p_0 = -H$ is not useful until we write the Hamiltonian in terms of the phase space coordinates and time: $H = H(p_i, x^j, t)$. We could do this by writing $L = p_\mu dx^\mu/dt$ in terms of x^i and dx^i/dt, but it is simpler to write p_0 directly in terms of (p_i, x^j, t). How?

A hint is given by the fact that in abandoning the affine parameterization by τ, we don't obtain the normalization of the four-momentum (eq. 11) automatically. Therefore we must add it as a constraint to the action of equation (13). We wish to use the energy integral $H_2 = -\frac{1}{2}m^2$ to reduce the order of the system (eqs. 10). Solving this relation for $-p_0$ in terms of the other variables yields the Hamiltonian on our reduced (6-dimensional) phase space.

For this procedure to be valid, it has to be shown that extremizing S_3 with respect to all possible phase space trajectories $\{p_i(t), x^i(t)\}$ is equivalent to extremizing S_2 with respect to $\{x^i(\tau), t(\tau)\}$ for τ being an affine parameter. Equivalently, we must show that solutions of equations (9) are solutions of equations (9) and vice versa. A proof is presented in Section 4.2 below.

Before presenting the technicalities, we state the key result of these notes, the Hamiltonian on our six-dimensional phase space $\{p_i, x^j\}$, obtained by solving $H_2(p_i, p_0, x^j, t) = -\frac{1}{2}m^2$ for $p_0 = -H$:

$$H(p_i, x^j, t) = -p_0 = \frac{g^{0i}p_i}{g^{00}} + \left[\frac{(g^{ij}p_ip_j + m^2)}{-g^{00}} + \left(\frac{g^{0i}p_i}{g^{00}}\right)^2\right]^{1/2} . \quad (14)$$

Note that here, as in the covariant case, the conjugate momenta are given by the (here, spatial) components of the momentum one-form. The inverse metric components $g^{\mu\nu}$ are, in general, functions of x^i and t. Equation (14) is exact; no approximation to the metric has been made. We only require that t be timelike, i.e. $g_{00} < 0$, in order to parameterize timelike geodesics.

The next section presents mathematical material that is optional for 8.962. However, it is recommended for those students prepared to explore differential geometry somewhat further. The application to Hamiltonian mechanics should help the student to better understand the mathematics of general relativity.

4 Hamiltonian mechanics and symplectic manifolds

The proof that the 8-dimensional phase space may be reduced to the six spatial dimensions while retaining a Hamiltonian description becomes straightforward in the context of symplectic differential geometry (see Section 4.2 below). Classical Hamiltonian mechanics is naturally expressed using differential forms and exterior calculus (Arnold 1989; see also Exercise 4.11 of Misner et al 1973). We present an elementary summary here, both to provide background for the proof to follow and to elucidate differential geometry through its use in another context. In fact, we are not ignoring general relativity but extending it; the Hamiltonian mechanics we develop is fully consistent with general relativity.

The material presented in this section is mathematically more advanced than Schutz (1985). Treatments may be found in Misner et al (1973, Chapter 4), Schutz (1980), Arnold (1989), and, briefly, in Appendix B of Wald (1984) and Carroll (1997).

We begin with the configuration space of a mechanical system of n degrees of freedom characterized by the generalized coordinates q^i (which may, for example, be the four spacetime coordinates of a single particle's worldline, or the three spatial coordinates only). The configuration space is a manifold V whose tangent space $TV_{\mathbf{q}}$ at each point $\mathbf{q}$ in the manifold is given by the set of all generalized velocity vectors $d\vec{q}/dt$ at $\mathbf{q}$. Note that t is any parameter for a curve $\mathbf{q}(t)$; we are not restricting ourselves to Newtonian mechanics with its absolute time.

The union of all tangent spaces at all points of the manifold is called the tangent bundle, denoted TV. The set TV has the structure of a manifold of dimension $2n$. There exists a differentiable function on TV, the Lagrangian, whose partial derivatives with respect to the velocity vector components defines the components of a one-form, the canonical momentum:

$$\tilde{p} \equiv \frac{\partial L}{\partial(d\vec{q}/dt)} \ . \tag{15}$$

To see that this is a one-form, we note that it is a linear function of a tangent vector: $\tilde{p}(d\vec{q}) = p_i dq^i$ is a scalar. At each point in the configuration space manifold, the set of all $\tilde{p}$ defines the cotangent space $T^*V_{\mathbf{q}}$. (The name cotangent is used to distinguish the dual space of one-forms from the space of vectors.)

The union of all cotangent spaces at all points of the manifold is called the cotangent bundle, T^*V. Like the tangent bundle, the cotangent bundle is a manifold of dimension $2n$. A point of T^*V is specified by the coordinates (p_i, q^j). The cotangent bundle is well known: it is phase space.

Having set up the phase space, we now discard the original configuration space V, its tangent vector space $TV_{\mathbf{q}}$ and the tangent bundle TV. To emphasize that the phase space is a manifold of dimension $2n$, we will denote it M^{2n} rather than by T^*V.

Being a manifold, the phase space has a tangent space of vectors. Each parameterized curve $\gamma(t)$ in phase space has, at each point in the manifold, a tangent vector $\vec{\xi}$ whose coordinate components are the $2n$ numbers $(dp_i/dt, dq^j/dt)$. The phase space also has one-forms, or linear functions of vectors. For example, the gradient of a scalar field $H(p_i, q^j)$ in phase space is a one-form. However, it will prove convenient to denote the gradient of a scalar using a new notation, the exterior derivative: $\mathbf{d}H \equiv \tilde{\nabla}H$. In the coordinate basis, $\mathbf{d}H$ has components $(\partial H/\partial p_i, \partial H/\partial q^j)$. In this section, forms will be denoted with boldface symbols.

One must be careful not to read too much into the positions of indices: $\partial H/\partial p_i$ and $\partial H/\partial q^i$ are both components of a one-form in phase space. They may also happen to be spacetime vectors and one-forms, respectively, but we are now working in phase space. In phase space, p_i and q^j have equal footing as coordinates. We will retain the placement of indices (i, j go from 1 to n) simply as a reminder that our momenta and position displacements may be derived from spacetime one-forms and vectors. This way we can arrive at physical equations of Hamiltonian dynamics that are tensor equations (hence valid for any coordinate system) in both spacetime and phase space.

As in spacetime, we define the basis one-forms by the gradient (here, the exterior derivative) of the coordinate fields: $\{\mathbf{d}p_i, \mathbf{d}q^j\}$. We can combine one-forms and vectors to produce higher-rank tensors through the operations of gradient and tensor product. It proves especially useful to define the antisymmetric tensor product, or wedge product. The wedge product of two one-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \equiv \boldsymbol{\alpha} \otimes \boldsymbol{\beta} - \boldsymbol{\beta} \otimes \boldsymbol{\alpha} \ . \tag{16}$$

The wedge product of two one-forms gives a 2-form, an antisymmetric $(0,2)$ tensor. The wedge product (tensor product with antisymmetrization) can be extended to produce p-forms with p less than or equal to the dimension of the manifold. A p-form is a fully antisymmetric, linear function of p vectors. Forms will be denoted by Greek letters.

Given a p-form $\boldsymbol{\alpha}$, we can obtain a $(p+1)$-form by exterior differentiation, $\mathbf{d}\boldsymbol{\alpha}$. Exterior differentiation consists of the gradient followed by antisymmetrization on all arguments. For p-form $\boldsymbol{\omega}^p$ and q-form $\boldsymbol{\omega}^q$, the exterior derivative obeys the relation

$$\mathbf{d}(\boldsymbol{\omega}^p \wedge \boldsymbol{\omega}^q) = \mathbf{d}\boldsymbol{\omega}^p \wedge \boldsymbol{\omega}^q + (-1)^p \boldsymbol{\omega}^p \wedge \mathbf{d}\boldsymbol{\omega}^q \ . \tag{17}$$

(Here p and q are integers having nothing to do with phase space coordinates.) Note that $\mathbf{dd}\boldsymbol{\omega} = 0$ for any form $\boldsymbol{\omega}$. Any form $\boldsymbol{\omega}$ for which $\mathbf{d}\boldsymbol{\omega} = 0$ is called a closed form.

Forms are most widely used to provide a definition of integration free from coordinates and the metric. Consider, for example, the line integral giving the work done by a force, $\int \vec{F} \cdot d\vec{x}$. If the force were a one-form $\boldsymbol{\theta}$ instead of a vector, and if $\vec{\xi}$ were the tangent vector to a path γ ($\vec{\xi} = d\vec{x}/dt$ where t parameterizes the path), we could write the work as $\int_\gamma \boldsymbol{\theta}(\vec{\xi})$ or $\int_\gamma \boldsymbol{\theta}$ for short. No coordinates are involved until we choose a coordinate basis, and no metric is required because we integrate a one-form instead of a vector with a dot product.

Similarly, a 2-form may be integrated over an orientable 2-dimensional surface. Integration is built up by adding together the results from many small patches of the surface. An infinitesimal patch may be taken to be the parallelogram defined by two tangent vectors, $\vec{\xi}$ and $\vec{\eta}$. The integral of the 2-form $\boldsymbol{\omega}$ over the surface σ is $\int_\sigma \boldsymbol{\omega}(\vec{\xi}, \vec{\eta})$ or $\int_\sigma \boldsymbol{\omega}$ for short.

The spacetime manifold received additional structure with the introduction of the metric, a $(0,2)$ tensor used to give the magnitude of a vector (and to distinguish timelike, spacelike and null vectors). A manifold with a positive-definite symmetric $(0,2)$ tensor defining magnitude is called a Riemannian manifold. When the eigenvalues of the metric have mixed signs (as in the case of spacetime), the manifold is called pseudo-Riemannian.

Phase space has no metric; there is no concept of distance between points of phase space. It has a special antisymmetric $(0,2)$ tensor instead, in other words a 2-form. We will call this fundamental form the **symplectic form** $\boldsymbol{\omega}$; Arnold (1989) gives it the cumbersome name "the form giving the symplectic structure." In terms of the coordinate basis one-forms $\mathbf{d}p_i$ and $\mathbf{d}q^j$, the symplectic form is

$$\boldsymbol{\omega} \equiv \mathbf{d}p_i \wedge \mathbf{d}q^i = \mathbf{d}p_1 \wedge \mathbf{d}q^1 + \cdots + \mathbf{d}p_n \wedge \mathbf{d}q^n \ . \tag{18}$$

Note the implied sum on paired upper and lower indices.

One of the uses of the metric is to map vectors to one-forms; the symplectic form fulfills the same role in phase space. Filling one slot of $\boldsymbol{\omega}$ with a vector yields a one-form, $\boldsymbol{\omega}(\cdot, \vec{\xi})$. It is easy to show that this mapping is invertible by representing $\boldsymbol{\omega}$ in the coordinate basis and showing that it is an orthogonal matrix. Therefore, every one-form has a corresponding vector.

There is a particular one-form of special interest in phase space, $\mathbf{d}H$ where $H(p, q, t)$ is the Hamiltonian function. The corresponding vector is the phase space velocity, i.e. the tangent to the phase space trajectory:

$$\begin{aligned} \boldsymbol{\omega}(\cdot, \vec{\xi}) &= \mathbf{d}p_i(\cdot)\mathbf{d}q^i(\vec{\xi}) - \mathbf{d}q^i(\cdot)\mathbf{d}p_i(\vec{\xi}) = \frac{dq^i}{dt}\mathbf{d}p_i - \frac{dp_i}{dt}\mathbf{d}q^i \\ &= \mathbf{d}H(\cdot) = \frac{\partial H}{\partial p_i}\mathbf{d}p_i + \frac{\partial H}{\partial q^i}\mathbf{d}q^i \ . \end{aligned} \tag{19}$$

Equating terms, we see that Hamilton's equations are given concisely by $\boldsymbol{\omega}(\vec{\xi}) = \mathbf{d}H$.

Besides giving the antisymmetric relationship between coordinates and momenta apparent in Hamilton's equations, the symplectic form allows us to define canonical transformations of the coordinates and momenta. The phase space components (p_i, q^j) transform with a $2n \times 2n$ matrix Λ to $(p_{\bar{i}}, q^{\bar{j}})$. A canonical transformation is one that leaves the symplectic form invariant. In matrix notation, this implies $\Lambda^T \omega \Lambda = \omega$. Thus, canonical invariance of a Hamiltonian system is analogous to Lorentz invariance in special relativity, where the transformations obey $\Lambda^T \eta \Lambda = \eta$ where η is the Minkowski metric.

The standard results of Hamiltonian mechanics are elegantly derived and expressed using the language of symplectic differential geometry. For example, Arnold (1989, §38 and §44D) shows that transformation of phase space induced by Hamiltonian evolution is canonical. This implies that the phase space area (the integral of $\boldsymbol{\omega}$, a 2-form) is preserved by Hamiltonian evolution. It is easy to show that not only $\boldsymbol{\omega}$ but also $\boldsymbol{\omega}^2 \equiv \boldsymbol{\omega} \wedge \boldsymbol{\omega}$ is a canonical invariant, as is $\boldsymbol{\omega}^p \equiv \boldsymbol{\omega} \wedge \cdots \wedge \boldsymbol{\omega}$ with p factors of $\boldsymbol{\omega}$, for all $p \leq n$. (Antisymmetry limits the rank of a p-form to $p \leq n$.) Thus, phase space volume is preserved by Hamiltonian evolution (Liouville theorem).

4.1 Extended phase space

Inspired by relativity, we can absorb the time parameter into the phase space to obtain a manifold of $2n + 1$ dimensions, denoted M^{2n+1} and called extended phase space. As we will see, this extension allows a concise derivation of the extremal form of the action under Hamiltonian motion.

Before proceeding, we should emphasize that the results of the previous section are not limited to nonrelativistic systems. Indeed, they apply to the phase space (p_μ, x^ν) of a single particle in general relativity where the role of time is played by the affine parameter τ. The relativistic Hamilton's equations (9) follow immediately from equation (19). Nonetheless, if we wish to parameterize trajectories by coordinate time (as we must for a system of more than one particle), we must show the consistency of the space-time split apparent in equation (14). We can do this by re-uniting coordinates and time in M^{2n+1}.

In M^{2n}, the symplectic form $\mathbf{d}p_i \wedge \mathbf{d}q^i$ is the fundamental object. In M^{2n+1}, we must incorporate the one-form $\mathbf{d}t$. This is done with a new one-form, the **integral invariant of Poincaré-Cartan**:

$$\boldsymbol{\omega} \equiv p_i \mathbf{d}q^i - H(p_i, q^j, t)\mathbf{d}t \ . \tag{20}$$

(The reader must note from context whether $\boldsymbol{\omega}$ refers to this one-form or to the symplectic 2-form.) This one-form looks deceptively like the integrand of the action, or the Lagrangian. However, it is a differential form on the extended phase space, not a function. Once we integrate it over a curve γ in M^{2n+1}, however, we get the action:

$$S = \int_\gamma \boldsymbol{\omega} = \int_A^B \left[p_i dq^i - H(p_i, q^j, t)dt\right] \ . \tag{21}$$

The integration is taken from A to B in the extended phase space.

Now suppose we integrate $\boldsymbol{\omega}$ from A to B along two slightly different paths and take the difference to get a close loop integral. To evaluate this integral we can use Stokes' theorem. In the language of differential forms, Stokes' theorem is written (Misner et al 1973, Chapter 4, or Wald 1984, Appendix B)

$$\int_{\partial M} \boldsymbol{\omega} = \int_M \mathbf{d}\boldsymbol{\omega} \tag{22}$$

Here, M is a p-dimensional compact orientable manifold with boundary ∂M and $\boldsymbol{\omega}$ is a $(p-1)$-form; $\mathbf{d}\boldsymbol{\omega}$ is its exterior derivative, a p-form. Note that M can be a submanifold of a larger space, so that Stokes' theorem actually implies a whole set of relations including the familiar Gauss and Stokes laws of ordinary vector calculus.

Applying equation (22) to the difference of actions computed along two neighboring paths with (q^i, t) fixed at the endpoints and using equation (17), we get

$$\delta S = \int_\sigma \mathbf{d}\boldsymbol{\omega} = \int_\sigma \mathbf{d}p_i \wedge \mathbf{d}q^i - \mathbf{d}H \wedge \mathbf{d}t \ , \tag{23}$$

where σ denotes the surface area in the extended phase space bounded by the two paths from A to B. Note the emergence of the fundamental symplectic form on M^{2n}.

Now, let us express the integrand of equation (23) in the coordinate basis of one-forms in M^{2n+1}, evaluating one of the vector slots using the tangent vector $\vec{\xi}$ to one of the two curves from A to B. The result is similar to equation (19):

$$\mathbf{d}\boldsymbol{\omega}(\cdot,\vec{\xi}\,) = \left(\frac{dq^i}{dt} - \frac{\partial H}{\partial p_i}\right)\mathbf{d}p_i + \left(-\frac{dp_i}{dt} - \frac{\partial H}{\partial q^i}\right)\mathbf{d}q^i + \left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right)\mathbf{d}t \ . \qquad (24)$$

The principal of least action states that $\delta S = 0$ for small variations about the true path, with (q^i, t) fixed at the end points. This will be true, for arbitrary small variations, if and only if $\mathbf{d}\boldsymbol{\omega}(\cdot,\vec{\xi}\,) = 0$ for the tangent vector along the extremal path. From equation (24), Hamilton's equations follow.

The solution of Hamilton's equations gives an extended phase-space trajectory with tangent vector $\vec{\xi}$ being an eigenvector of the 2-form $\mathbf{d}\boldsymbol{\omega}$ with zero eigenvalue. Arnold (1989) proves that, for any differentiable function H defined on M^{2n+1}, the two-form $\mathbf{d}\boldsymbol{\omega}$ has exactly one eigenvector with eigenvalue zero, $(\partial H/\partial p_i, -\partial H/\partial q^i, 1)$. This is a vector field in M^{2n+1} and it defines a set of integral curves (field lines, to which it is tangent) called the "vortex lines" of the one-form $\boldsymbol{\omega}$. The vortex lines are precisely the trajectories of Hamiltonian flow, i.e. the solutions of equations (12).

A bundle of vortex lines is called a vortex tube. From Stokes' theorem, the circulation of a vortex tube, defined as the integral of the Poincaré-Cartan integral invariant around a closed loop bounding the vortex tube, is an integral of motion. (This is why $\boldsymbol{\omega}$ is called an integral invariant.) If the bounding curves are taken to lie on hypersurfaces of constant time, it follows that $\oint p_i \mathbf{d}q^i$ is also an integral of motion. By Stokes' theorem, this implies that the fundamental form $\mathbf{d}p_i \wedge \mathbf{d}q^i$ is an integral invariant. Thus, Hamiltonian evolution is canonical and preserves phase space areas and volumes.

By adding t to our manifold we have partially unified coordinates and time. Can we go all the way to obtain a spacetime covariant formulation of Hamiltonian dynamics? For the case of single particle motion, the answer is clearly yes. If we write $H = -p_0$ and $t = q^0$, the integral invariant of Poincaré-Cartan takes the simple form $\boldsymbol{\omega} = p_\mu \mathbf{d}q^\mu$ where μ takes the range 0 to n. Now $\mathbf{d}\boldsymbol{\omega}$ looks like the symplectic form on M^{2n+2}, except that here p_0 is not a dynamical variable but rather a function on M^{2n+1}. However, we can promote it to the status of an independent variable by defining a new Hamiltonian $H'(p_\mu, q^\nu)$ on M^{2n+2} such that $H' =$ constant can be solved for p_0 to give $-p_0 = H(p_i, q^j, q^0 = t)$. A simple choice is $H' = p_0 + H$.

Having subsumed the parameter for trajectories into the phase space, we must introduce a new parameter, τ. Because $\partial H'/\partial\tau = 0$, the solution of Hamilton's equations in M^{2n+2} will ensure that H' is a constant of motion. This is exactly what happened with the relativistically covariant Hamiltonian H_2 in Section 2 (eqs. 8 and 11).

The reader may now ask, if the Hamiltonian is independent of time, is it possible to reduce the dimensionality of phase space by two? The answer is yes; the next section shows how.

4.2 Reduction of order

Hamilton's equations imply that when $\partial H/\partial t = 0$, H is an integral of motion. In this case, phase space trajectories in M^{2n} are confined to the $(2n-1)$-dimensional hypersurface $H =$ constant. This condition may be used to eliminate t and choose one of the coordinates to become a new "time" parameter, with a new Hamiltonian defined on the reduced phase space.

This procedure was used in Section 3 to reduce the relativistically covariant 8-dimensional phase space $\{p_\mu, x^\nu\}$ with Hamiltonian given by equation (8) to the 6-dimensional phase space $\{p_i, x^j\}$ with the Hamiltonian of equation (14). While this reduction is plausible, it remains to be proved that the reduced phase space is a symplectic manifold and that the new Hamiltonian is given by the momentum conjugate to the time coordinate. The proof is given here.

Starting from the conserved Hamiltonian $H(p,q) \equiv H(p_0, p_i, q^0, q^j) = h$ with $1 \le i, j \le n-1$, we assume that (in some region) this equation can be solved for the momentum coordinate p_0:

$$p_0 = -K(P_i, Q^j, T; h) \tag{25}$$

where $P_i = p_i$, $Q^i = q^i$, and $T = q^0$. Note that any of the coordinates may be eliminated, with its conjugate momentum becoming (minus) the new Hamiltonian. Thus, the reduction of order is compatible with relativistic covariance. However, it can be applied to any Hamiltonian system, relativistic or not.

Next we write the integral invariant of Poincaré-Cartan in terms of the new variables:

$$\boldsymbol{\omega} = p_0 \mathbf{d}q^0 + p_i \mathbf{d}q^i - H\mathbf{d}t = P_i \mathbf{d}Q^i - K\mathbf{d}T - \mathbf{d}(Ht) + t\mathbf{d}H \ . \tag{26}$$

Recall that this is a one-form defined on M^{2n+1}.

Now let γ be an integral curve of the canonical equations (12) lying on the $2n$-dimensional surface $H(p,q) = h$ in the $(2n+1)$-dimensional extended phase space $\{p,q,t\}$. Thus, γ is a vortex line of the two-form $p\mathbf{d}q - H\mathbf{d}t = p_0\mathbf{d}q^0 + p_i\mathbf{d}q^i - H\mathbf{d}t$. We project the extended phase space M^{2n+1} onto the phase space $M^{2n} = \{p,q\}$ by discarding the time parameter t. The surface $H = h$ projects onto a $(2n-1)$-dimensional manifold M^{2n-1} with coordinates $\{P_i, Q^j, T\}$. Discarding t, the integral curve γ projects onto a curve $\bar{\gamma}$ also in M^{2n-1}.

The coordinates $(P_i, Q^j, T) = (p_i, q^j, q^0)$ locally (and perhaps globally) cover the submanifold M^{2n-1} (the surface $H =$ constant in $M^{2n} = \{p,q\}$). We now show that M^{2n-1} is the extended phase space for a symplectic manifold with Hamiltonian K.

We do this by examining equation (26) while noting that the integral curve γ lies on the surface $H =$ constant. Clearly the last term in equation (26) vanishes on M^{2n-1}. Next, $\mathbf{d}(Ht)$ does not affect the vortex lines of $\boldsymbol{\omega}$ because $\mathbf{dd}(Ht) = 0$. (The variation of the action is invariant under the addition of a total derivative to the Lagrangian.) But the vortex lines of $P_i\mathbf{d}Q^i - K\mathbf{d}T$ satisfy Hamilton's equations (Sect. 4.1). Thus we have proven that reduction of order preserves Hamiltonian evolution.

The solution curves $\bar{\gamma}$ on M^{2n-1} are vortex lines of $p\mathbf{d}q = P\mathbf{d}Q - K\mathbf{d}T$. Thus, they are extremals of the integral $\int p\mathbf{d}q$. In other words, if the Hamiltonian function $H(q,p)$ in M^{2n+1} is independent of time, then the phase space trajectories satisfying Hamilton's equations are extremals of the integral $\int p\mathbf{d}q$ in the class of curves lying on M^{2n-1} with fixed endpoints of integration. The converse is also true (Arnold 1989): if $\partial H/\partial t = 0$, the extremals of the "reduced action"

$$\int_\gamma p\mathbf{d}q = \int_\gamma \frac{\partial L}{\partial(d\vec{q}/d\tau)}(\tau)\frac{d\vec{q}}{d\tau}\,d\tau \tag{27}$$

with fixed endpoints, $\delta q = 0$, are solutions of Hamilton's equations in M^{2n+1}. This is known as Maupertuis' principle of least action. Note that the principle can only be implemented if p_i is expressed as a function of q and $\dot{q}$ so that the integral is a functional of the configuration space trajectory. Also, because the time parameterization is arbitrary, Maupertuis' principle determines the shape of a trajectory but not the time (t does not appear in eq. 27); in order to determine the time we must use the energy integral.

These results justify the approach of Section 3. The spacetime trajectories are extremals of equation (13) as a consequence of $\partial H_2/\partial\tau = 0$ (eq. 8) and Maupertuis' principle. The order is reduced further by using $H_2 = -\frac{1}{2}m^2$ to solve for $-p_0$ as the new Hamiltonian $H(p_i, x^j, t)$, equation (14).

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