Computational Tools for the Safety Control of a Class of Piecewise Continuous Systems with Imperfect Information on a Partial Order

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Abstract

This paper addresses the two-agent safety control problem for piecewise continuous systems with disturbances and imperfect state information. In particular, we focus on a class of systems that evolve on a partial order and whose dynamics preserve the ordering. While the safety control problem with imperfect state information is prohibitive for general classes of nonlinear and hybrid systems, the class of systems considered in this paper admits an explicit solution. We compute this solution with linear complexity discrete-time algorithms that are guaranteed to terminate. The proposed algorithms are illustrated on a two-vehicle collision avoidance problem and implemented on a hardware roundabout test-bed.

1   Introduction

In this paper, we consider a class of piecewise continuous systems that evolve on a partial order and propose an explicit solution to the two-agent safety control problem with imperfect state information.

There has been a wealth of research on safety control for general nonlinear and hybrid systems assuming perfect state information [37, 43, 46, 47]. In these works, the safety control problem is elegantly formulated in the context of optimal control and leads to the Hamilton-Jacobi-Bellman (HJB) equation. This equation implicitly determines the maximal controlled invariant set and the least restrictive feedback control map. Due to the complexity of exactly solving the HJB equation, researchers have been investigating approximated algorithms for computing inner-approximations of the maximal controlled invariant set [29, 30, 41, 47]. Termination of the algorithm that computes the maximal controlled invariant set is often an issue and work has been focusing on determining special classes of systems that allow one to prove termination (see [43] and the references therein). The safety

control problem for hybrid systems has also been investigated within a viability theory approach by a number of researchers (see [12, 25, 26], for example).

The above cited works focus on control problems with full state information and, as a result, static feedback control maps are designed. When the state of the system is not fully available for control, the above approaches cannot be applied. The advances in state estimation for hybrid systems of the past few years [5, 13–15, 18, 23, 42, 50] have set the basis for the development of dynamic feedback (state estimation plus control) for hybrid systems [20, 21, 51]. In particular, [51] proposes a solution to the control problem with imperfect state information for rectangular hybrid automata that admit a finite-state abstraction. For this case, the problem is shown to have exponential complexity in the size of the system. This problem is solved by determining the maximal controlled invariant safe set, that is, the set of all initial information states for which a dynamic control law exists guaranteeing that the current information state never intersects the set of bad states. Since the information state is a set, the maximal controlled invariant set is a *set of sets*, making its computation even harder than for the static feedback problem. As a consequence, for general hybrid systems the dynamic feedback problem under safety specifications is prohibitive. Dynamic feedback in a special class of hybrid systems with imperfect discrete state information is presented in [20], however the problem of computing the maximal controlled invariant set is not considered. Dynamic control of block triangular order preserving hybrid automata under imperfect continuous state information is considered in [21] for discrete-time systems, and an algorithm for computing an inner approximation of the maximal controlled invariant set is proposed. Dynamic feedback for order preserving systems in continuous time is considered in [22, 27]. However, in [22] only a cooperative game structure is considered and in [27] only a competitive game structure is addressed. In [49], dynamic feedback is addressed for a class of hybrid automata with imperfect state information.

Since for general classes of hybrid systems, the dynamic feedback problem is prohibitive, we consider this problem in a restricted class of hybrid systems, which is still general enough to model application scenarios of interest. In particular, we focus on a class of hybrid systems whose state and input spaces have a partial ordering and generate trajectories that preserve this ordering. The problem is posed as an order preserving game structure, which is an approach that unifies the special cases of cooperative [22] and competitive [27] game structure between two agents in a general framework. By exploiting the order preserving property of the flow, we obtain an explicit solution for the maximal controlled invariant set and for a dynamic control map. We show that the static and dynamic feedback problems are solved by the same control map, which is computed on the state in the first case and on a state estimate in the second case. This implies separation between state estimation and control for the class of systems considered. For safety control problems generated by a specific conflict topology, this solution can be computed in discretetime by linear complexity algorithms, for which we can show termination.

Dynamical systems whose flow preserves an ordering on the state space with respect to state and input are called monotone control systems [6]. Monotone control systems have received considerable attention in the dynamical systems and control literature as several biological processes involving competing or cooperating species are monotone [45]. More general bio-molecular systems can be modeled as the interconnection of monotone control systems [7, 8, 24]. There are also a large number of engineering applications that feature agents evolving on partial orders with order preserving dynamics. Multi-robot systems engaged in target assignment tasks have been shown to evolve according to an order preserving dynamics on the partial order established on the set of all possible assignments [23]. Railway control networks feature a number of agents (the trains) that evolve on
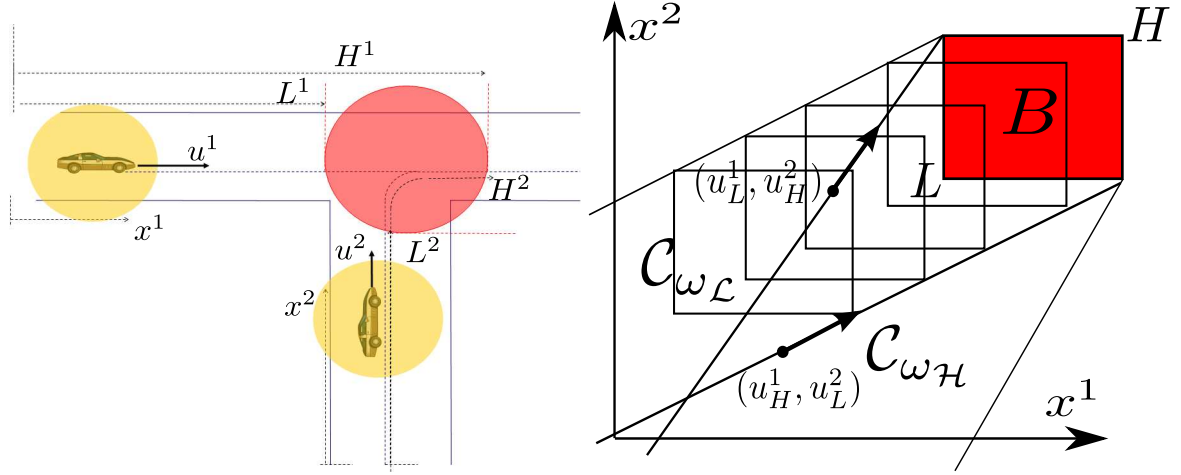
Figure 1: (Left) Vehicles approaching a "T" intersection. (Right) The sets C_{ω_L} and C_{ω_H}, in which $L = (L^1, L^2)$, $H = (H^1, H^2)$, and $B =]L^1, H^1[\times]L^2, H^2[$.

pre-defined paths (the railways) unidirectionally according to the Lomonossoff's model, which is an order preserving system on the path [31, 39]. Transportation networks also feature vehicles traveling unidirectionally on their paths and lanes, which impose an ordering on their motion. In air traffic networks, the longitudinal motion of each aircraft along its prescribed route can also be modeled by an order preserving dynamics [32, 40].

In this paper, we illustrate the application of the proposed technique to a two-vehicle collision avoidance problem as found in traffic intersections or modern roundabouts in the presence of modeling uncertainty, missing communication, and imperfect state information.

Motivating Example. Consider the problem of preventing a collision between two vehicles approaching an intersection as depicted in the left panel of Figure 1. A collision occurs if the two vehicles are in the shaded red area B at the same time. The problem is to design a controller that guarantees that the vehicles do not collide excluding the trivial solution in which the vehicles stop. In general, the vehicle states can be subject to large uncertainties as deriving from GPS, for example, and the dynamic model can be affected by modeling error. For the sake of explaining the basic idea of our solution, consider the case in which the dynamics of vehicles 1 and 2 are given by $\dot{x}^1 = u^1$, $\dot{x}^2 = u^2$, respectively, with $u^1, u^2 \in [u_L, u_H]$ and $u_L, u_H > 0$. A more realistic second order hybrid model for each of the vehicle dynamics will be considered in the simulation section. Assume also perfect information of the state (x^1, x^2). Here, x^1 and x^2 denote the longitudinal displacements of the vehicles on their paths as shown in the figure. In this coordinate system, $B =]L^1, H^1[\times]L^2, H^2[$. To solve the control problem, we seek to compute the set of all initial conditions $(x^1(0), x^2(0))$ that are taken to B for all inputs (u^1, u^2). This set is called the capture set, denoted C, and is the complement of the largest controlled invariant set that does not contain B. On the basis of the capture set, we then seek to design a feedback map that guarantees that any state starting outside C is kept outside C.

This general problem can be elegantly formulated as an optimal control problem with terminal cost, which leads to an implicit solution expressed as the solution of a PDE [37]. In this example, however, there is a rich structure that can be exploited to obtain an immediate explicit solution without the need of solving an optimal control problem. In particular, the dynamics of each vehicle preserve the standard ordering on $\mathbb{R}$, that is, higher initial conditions $x^i(0)$ and higher inputs u^i lead to higher values of the state $x^i(t)$ for all time. Denote by C_{ω_H} the set of all initial conditions that are taken to B when

the input to the system is set to $\omega_{\mathcal{H}} := (u_L, u_H)$, that is, vehicle 1 applies constant $u^1 = u_L$ and vehicle 2 applies constant $u^2 = u_H$. Similarly, denote by $C_{\omega_{\mathcal{L}}}$ the set of all initial conditions that are taken to B when the input to the system is set to $\omega_{\mathcal{L}} := (u_H, u_L)$, that is, vehicle 1 applies constant $u^1 = u_H$ and vehicle 2 applies constant $u^2 = u_L$. Because the dynamics of the system have the order preserving properties described above, one can show that the capture set is given by the intersection of these two sets, that is, $C = C_{\omega_{\mathcal{L}}} \cap C_{\omega_{\mathcal{H}}}$ (right panel of Figure 1). In practice, this means the following. The state x is taken to B for all input choices if and only if it is taken to B both when (a) vehicle 1 applies maximum control and vehicle 2 applies minimum control and (b) vehicle 1 applies minimum control and vehicle 2 applies maximum control.

The relevance of having $C = C_{\omega_{\mathcal{L}}} \cap C_{\omega_{\mathcal{H}}}$ resides in the following key points. First, $C_{\omega_{\mathcal{L}}}$ and $C_{\omega_{\mathcal{H}}}$ can be easily computed by backward integration without the need of optimizing over the control values because the controls are fixed. Second, this backward integration task can be performed by simply propagating back through the dynamics the lower and upper bounds of B, that is, L and H, respectively, with fixed inputs (refer to the right panel of Figure 1). In discrete-time, this can be performed by a linear complexity algorithm. Furthermore, checking whether a state (x^1, x^2) belongs to either $C_{\omega_{\mathcal{L}}}$ or $C_{\omega_{\mathcal{H}}}$ can be performed in finite time because the backward integration of L and H leads to strictly decreasing sequences: once the decreasing sequences starting in H passes beyond the point (x^1, x^2), one has enough information to establish whether (x^1, x^2) belongs either to $C_{\omega_{\mathcal{L}}}$ or to $C_{\omega_{\mathcal{H}}}$. Finally, a feedback map is one that imposes the control $\omega_{\mathcal{H}} = (u_L, u_H)$ when the state is inside $C_{\omega_{\mathcal{H}}}$ and on the boundary of $C_{\omega_{\mathcal{L}}}$, while it imposes the control $\omega_{\mathcal{L}} = (u_H, u_L)$ when the state is inside $C_{\omega_{\mathcal{L}}}$ and on the boundary of $C_{\omega_{\mathcal{H}}}$ (right panel of Figure 1). This way, we provide also a closed form solution for the feedback map. *In this paper, we show that this basic result holds for arbitrary order preserving dynamics, for the case in which these dynamics are affected by disturbances, and when only imperfect state information is available.*

This paper is organized as follows. In Section 2, we introduce basic definitions and the class of systems that we consider is introduced in Section 3. In Section 4, we provide a mathematical statement of the safety control problem. In Section 5, we give the main result of the paper, namely the computation of the maximal controlled invariant set and the related dynamic feedback control map. In Section 6, we present a discrete-time algorithm for computing the maximal controlled invariant set and the dynamic feedback map. In Section 7, we present an example application involving a two-vehicle collision avoidance problem at a traffic intersection. Several of the proofs are found in the Appendix.

2 Notation and Basic Definitions

For the set $A \subset X$ with X a normed vector space, denote the complement $\sim A := X \backslash A$, the interior $\mathring{A}$, the closure $\overline{A}$, the closed convex hull $\overline{\mathrm{co}}\, A$, the boundary ∂A, and the set of all subsets contained in A by 2^A. For $x \in \mathbb{R}^n$, denote the Euclidean norm $\|x\|$, and the inner product $\langle y \,|\, x \rangle$. For $x \in \mathbb{R}^n$ and set $A \subset \mathbb{R}^n$, denote the distance from x to A as $d(x, A) := \inf_{y \in A} \|x - y\|$. This extends to the distance between two sets $A, B \in \mathbb{R}^n$, where $d(A, B) := \inf_{y \in A} d(y, B)$. Let $]a, b[,\ ]a, b],\ [a, b] \subset \mathbb{R}$ denote the open, half open, and closed intervals respectively. This notation extends to interval sets $]a, b[,\ ]a, b],\ [a, b] \subset \mathbb{R}^n$, where, for example, $]a, b[:=]a_1, b_1[\times \ldots \times]a_n, b_n[$. The open ball of radius $\epsilon > 0$ centered at $x \in \mathbb{R}^n$ is denoted $B(x, \epsilon) := \{z \in \mathbb{R}^n \mid \|x - z\| < \epsilon\}$. For the set $A \subset \mathbb{R}^n$, we define an open neighborhood about A of radius $\epsilon > 0$ by $B(A, \epsilon) := \{z \in \mathbb{R}^n \mid d(z, A) < \epsilon\}$. Denote the canonical basis vectors $\hat{e}_i$ for $i \in \{1, 2, \ldots, n\}$. For $x \in \mathbb{R}^n$, denote the ith component by
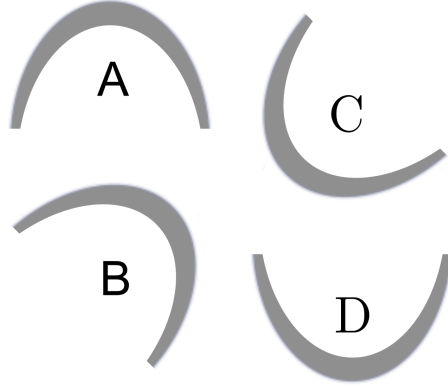
Figure 2: The sets $A, B \subset \mathbb{R}^2$ are o.p.c., while the sets $C, D \subset \mathbb{R}^2$ are not o.p.c.

$x_i := \langle x | \hat{e}_i \rangle$. Denote the canonical projection $\tau_i : \mathbb{R}^n \to \mathbb{R}$ defined by $\tau_i(x) = x_i$, which naturally extends to sets. Denote the unit sphere $\mathbb{S}^n$ and the unit disk $\mathbb{D}^n$, where $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $\mathbb{D}^{n+1} := \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$. For sets $A, B \subseteq \mathbb{R}^n$ we define the binary relation $A \prec B$ ($A \preceq B$) if $\tau_1(A) \cap \tau_1(B)$ is non-empty and for all $x \in A$ and $y \in B$ such that $x_1 = y_1$, we have $x_2 < y_2$ ($x_2 \leq y_2$).

Denote the space of n-times continuously differentiable functions from A into B as $C^n(A, B)$. We use the notation $F : A \rightrightarrows B$ to denote a set-valued map from A into B. For $A \subset X$ and $f : X \to Y$, we define the image of A under f as $f(A) := \{f(x) \in Y \mid x \in A\}$. We denote the space of piecewise continuous signals on A as $S(A) := PC(\mathbb{R}_+, A)$. Denote the unit interval $I := [0, 1]$. For the set $A \subset \mathbb{R}^2$, we will call a path $\gamma \in C^0(I, A)$ simple if γ is injective. We will call it closed if $\gamma(0) = \gamma(1)$. We define the Cone at vertex $x \in \mathbb{R}^n$ with respect to $a_1, a_2, \ldots, a_k \in \mathbb{R}^n$ as $\text{Cone}_{\{a_1, a_2, \ldots, a_k\}}(x) := \{y \in \mathbb{R}^n \mid \langle y - x | a_i \rangle \geq 0 \; \forall \; i \in \{1, 2, \ldots, k\}\}$. For $x \in \mathbb{R}^2$, we use the shorthand notation $\text{Cone}_+(x) := \text{Cone}_{\{\hat{e}_1, \hat{e}_2\}}(x) \subset \mathbb{R}^2$ and $\text{Cone}_-(x) := \text{Cone}_{\{-\hat{e}_1, -\hat{e}_2\}}(x) \subset \mathbb{R}^2$. We use the following continuity definition for set-valued maps [11].

Definition 1. For metric spaces A and D, a set-valued map $F : A \rightrightarrows D$ is said *upper hemi-continuous* at $x \in A$ if for all $\epsilon > 0$ there is $\eta > 0$ such that $F(y) \subset B(F(x), \epsilon)$ for all $y \in B(x, \eta)$.

We next introduce a set characterization useful in formulating safety control problems for order preserving systems.

Definition 2. A path $\gamma \in C^0(I, \mathbb{R}^2)$ is said to be *order preserving connected* (o.p.c.) if it is simple, and for all $x \in \mathbb{R}^2$ $\text{Cone}_+(x) \cap \gamma(I) \neq \emptyset$ implies that $\text{Cone}_+(x) \cap \gamma(I)$ is path connected. A set $D \subseteq \mathbb{R}^2$ is said o.p.c. if for all $x, y \in D$, there exists a $\gamma \in C^0(I, D)$ such that $\gamma(0) = x, \gamma(1) = y$ and γ is o.p.c. (Figure 2).

Note that any convex set is trivially o.p.c. A partial order is a set P with a partial order relation "$\leq$", which we denote by the pair (P,$\leq$) [19]. In this paper, we are mostly concerned with the partial order $(\mathbb{R}^n, \leq)$ defined by component-wise ordering, that is, for all $w, z \in \mathbb{R}^n$ we have that $w \leq z$ if and only if $w_i \leq z_i$ for all $i \in \{1, 2, \ldots, n\}$. Given sets $A, B \subset \mathbb{R}^n$, we say $A \leq B$ if $a \leq b$ for all $a \in A$ and $b \in B$. For $U \subseteq \mathbb{R}^m$, we define the partial order $(S(U), \leq)$ by component-wise ordering for all time, that is, for all $\mathbf{w}, \mathbf{z} \in S(U)$ we have that $\mathbf{w} \leq \mathbf{z}$ provided $\mathbf{w}(t) \leq \mathbf{z}(t)$ for all $t \in \mathbb{R}_+$. Suppose (P,$\leq_P$) and (Q,$\leq_Q$) are two partially ordered sets. A map $f : P \to Q$ is an order preserving map provided $x \leq_P y$ implies $f(x) \leq_Q f(y)$.

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3   Class of Systems Considered

We consider piecewise continuous systems, with imperfect state information. This includes the set of hybrid systems with no continuous state reset and no discrete state memory, also referred to as switched systems [17].

Definition 3. A *piecewise continuous system* Σ with imperfect state information is a collection $\Sigma = (X, U, O, f, h)$, in which
- (*i*) $X \subset \mathbb{R}^n$ is a set of continuous variables;
- (*ii*) $U \subset \mathbb{R}^m$ is a set of continuous inputs;
- (*iii*) $O \subset X$ is a set of continuous outputs;
- (*iv*) $f : X \times U \to X$ is a *piecewise continuous* vector field;
- (*v*) $h : O \rightrightarrows X$ is an output map.

For an output measurement $z \in O$, the function $h(z)$ returns the set of all states compatible with the current output. We assume h is closed valued, that is, for all $z \in O$, $h(z)$ is closed. We assume that there is a $\bar{z} \in O$ such that $h(\bar{z}) = X$, corresponding to missing sensory information. We let $\phi(t, x, \mathbf{u})$ denote the flow of Σ at time $t \in \mathbb{R}_+$, with initial condition $x \in X$ and input $\mathbf{u} \in S(U)$ [35]. Denote the ith component of the flow by $\phi_i(t, x, \mathbf{u})$.

We restrict the class of piecewise systems to order preserving systems. These systems are defined on the partial orders $(\mathbb{R}^n, \leq)$ and $(S(U), \leq)$ as follows.

Definition 4. The system $\Sigma = (X, U, O, f, h)$ is an *order preserving* system provided there exist constants $u_L, u_H \in \mathbb{R}^m$ and $\xi > 0$ such that
- (*i*) $U = [u_L, u_H] \subset \mathbb{R}^m$;
- (*ii*)   The flow $\phi(t, x, \mathbf{u})$ is an order preserving map with respect to state x and input $\mathbf{u}$;
- (*iii*) $f_1(x, u) \geq \xi$ for all $(x, u) \in X \times U$;
- (*iv*)   For all $z \in O, h(z) = [\inf h(z), \sup h(z)] \subseteq \mathbb{R}^n$.

Conditions for establishing order preserving properties of the flow generated by a smooth vector field $f(x, u)$ have been previously addressed [6]. Sufficient conditions for establishing order preserving properties of piecewise-affine systems have been addressed in [9]. For systems in which x_1 is a position (as in the case of the example illustrated in Section 1), condition (iii) guarantees that the system never comes to a halt. More generally, it enforces the liveness of the system. Condition (iv) requires that the set $h(z)$ for any measurement z is an interval in the $(\mathbb{R}^n, \leq)$ partial order. We next define the parallel composition of two systems as defined in standard references [28].

Definition 5. For $\Sigma^1 = (X^1, U^1, O^1, f^1, h^1)$ and $\Sigma^2 = (X^2, U^2, O^2, f^2, h^2)$, we define the *parallel composition* $\Sigma = \Sigma^1 \| \Sigma^2 := (X, U, O, f, h)$, in which $X := X^1 \times X^2$, $U := U^1 \times U^2$, $O := O^1 \times O^2$, $f := (f^1, f^2)$ and $h := (h^1, h^2)$.

For $x = (x^1, x^2) \in X^1 \times X^2$ and $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) \in S(U^1 \times U^2)$, we denote the flow of the parallel composition $\Sigma^1 \| \Sigma^2$ as $\phi(t, x, \mathbf{u}) = (\phi^1(t, x^1, \mathbf{u}^1), \phi^2(t, x^2, \mathbf{u}^2))$ in which $\phi^1(t, x^1, \mathbf{u}^1) \in X^1$ and $\phi^2(t, x^2, \mathbf{u}^2) \in X^2$. We denote $\phi_j(t, x, \mathbf{u}) := (\phi_j^1(t, x^1, \mathbf{u}^1), \phi_j^2(t, x^2, \mathbf{u}^2))$.

We next define a new partial order $(S(U), \trianglelefteq)$ on input signals of the parallel composition of two systems as follows.

Definition 6. Given the parallel composition $\Sigma = \Sigma^1 \| \Sigma^2$, the input set $U = U^1 \times U^2$, and $\mathbf{u}, \mathbf{v} \in S(U)$, we say that $\mathbf{u} \trianglelefteq \mathbf{v}$ if $\mathbf{u}^1 \geq \mathbf{v}^1$ and $\mathbf{u}^2 \leq \mathbf{v}^2$.

Proposition 1. *Consider* $\Sigma = \Sigma^1 \| \Sigma^2$ *in which* Σ^1 *and* Σ^2 *are order preserving systems. For* $x \in X$ *and input signals* $\mathbf{u}, \mathbf{v} \in S(U)$ *such that* $\mathbf{u} \trianglelefteq \mathbf{v}$, *we have that* $\phi_1(\mathbb{R}_+, x, \mathbf{u}) \leq \phi_1(\mathbb{R}_+, x, \mathbf{v})$.

The proof is given in the Appendix. This proposition states that if two inputs satisfy the "$\trianglelefteq$" relation, the trajectories generated by these inputs (with the same initial condition) must satisfy the "$\leq$" relation, that is, one trajectory will always "lie above" the other in the (x_1^1, x_1^2) subspace.

4 Problem Formulation

In order to formulate the control problem, we first specify what inputs of $\Sigma = \Sigma^1 \| \Sigma^2$ are controlled and what are uncontrolled (disturbances). This is performed by introducing a two-player game structure on the parallel composition of the two systems as follows.

Definition 7. A *two-player piecewise continuous game structure* is a tuple $\mathscr{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$ in which

   (i) $\Sigma = \Sigma^1 \| \Sigma^2 = (X, U, \mathcal{O}, f, h)$ is the parallel composition of two piecewise continuous systems;

  (ii) $\Omega, \Delta \subset \mathbb{R}^m \times \mathbb{R}^m$ are the control and disturbance sets, respectively;

 (iii) $\varphi : \Omega \times \Delta \to U$ is the game input map;

 (iv) $\mathbf{B} \subset X$ is a set of bad states.

The disturbance $\delta \in \Delta$ and the control $\omega \in \Omega$ determine the input $u = (u^1, u^2)$ of Σ through the map φ, that is, we have that $u = \varphi(\omega, \delta)$. Extend the map φ to operate on signals by $\varphi(\omega, \delta) := \mathbf{u}$ where $\mathbf{u}$ is the signal such that $\mathbf{u}(t) = \varphi(\omega(t), \delta(t))$. We denote the flow of the game by $\phi(t, x, \varphi(\omega, \delta))$. We will say that the disturbance δ wins the game if the flow of $\mathscr{G}$ enters $\mathbf{B}$, while the controller ω wins the game if the flow of $\mathscr{G}$ never enters $\mathbf{B}$.

Definition 8. A game structure $\mathscr{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$ is an *order preserving* game structure provided

   (i) $\Sigma = \Sigma^1 \| \Sigma^2$ with Σ^1 and Σ^2 order preserving systems;

  (ii) $\Delta := [\delta_L^1, \delta_H^1] \times [\delta_L^2, \delta_H^2] := [\delta_L, \delta_H]$ and $\Omega := [\omega_L^1, \omega_H^1] \times [\omega_L^2, \omega_H^2] := [\omega_L, \omega_H]$;

 (iii) The game input $\varphi(\omega, \delta) = (\varphi^1(\omega^1, \delta^1), \varphi^2(\omega^2, \delta^2))$ is an order preserving map with respect to control ω and disturbance δ;

 (iv) $\mathbf{B} := \{x \in \mathbb{R}^n \times \mathbb{R}^n \mid (x_1^1, x_1^2) \in B\}$ with B an o.p.c. set.

The order preserving property of φ can be interpreted as follows. For the control signals $\omega, \mathbf{w} \in S(\Omega)$ and disturbance signals $\delta, \mathbf{d} \in S(\Delta)$, if we have that $\omega \leq \mathbf{w}$ and $\delta \leq \mathbf{d}$, then $\varphi(\omega, \delta) \leq \varphi(\mathbf{w}, \mathbf{d})$. Similarly, $\omega \trianglelefteq \mathbf{w}$ and $\delta \trianglelefteq \mathbf{d}$ implies $\varphi(\omega, \delta) \trianglelefteq \varphi(\mathbf{w}, \mathbf{d})$. The utility of this formulation lies in the ability to model cooperation and competition between two agents under a simple unified framework. For a cooperative scenario, in which both systems Σ^1 and Σ^2 are affected by the control but not by the disturbance, we let $\varphi_{\text{coop}}(\omega, \delta) := \omega$. For a competitive scenario, in which system Σ^2 is an adversary while system Σ^1 is completely controlled, we have $\varphi_{\text{comp}}(\omega, \delta) := (\omega^1, \delta^2)$. The more general case, in which both systems Σ^1 and Σ^2 are affected by control and disturbance, could represent model

uncertainty for example. An instance of each case is presented in Section 7. One can easily check that the example proposed in Section 1 is an order preserving game structure in which $\varphi = \varphi_{\text{coop}}$.

In the reminder of this paper, we assume (unless stated otherwise) that the flow of $\mathscr{G}$ is continuous with respect to initial condition, with respect to input, and with respect to time. Continuity conditions for the flow of a hybrid system have been previously investigated by, for example, [36] and the references therein. For the compact set of initial conditions $A \subset X$, we assume that the set-valued flow $\phi(t, A, S(U))$ is compact and upper hemi-continuous with respect to time. This property is satisfied, for example, in systems generated by the differential inclusion $\dot{x} \in f(x, U)$, in which $f(x, U)$ is a Marchaud map (see Theorem 3.5.2 in [10], and Corollary 4.5 in [44]). Note that, given a differential inclusion $\dot{x} \in f(x, U)$, the closed convex hull generates a differential inclusion $\dot{x} \in \overline{\text{co}} f(x, U)$, which is Marchaud provided that it is upper hemi-continuous and bounded above by some linear affine function, that is, $\|f(x, U)\| \leq c(\|x\| + 1)$. This allows for the over-approximation of a given system with another one that has the desired properties of the set-valued flow.

Given a game structure $\mathscr{G}$, we consider the problem of designing a controller that on the basis of the output information guarantees that the flow of $\mathscr{G}$ never enters the bad set of states $\mathbf{B}$ for all disturbance choices. For stating the control problem with imperfect state information, denote by $\hat{x}(t, \hat{x}_0, \omega, \mathbf{z})$ the set of all possible states at time t compatible with the set of initial conditions $\hat{x}_0 \subset X$ and measurable signals ω and $\mathbf{z}$. More formally,

$$\hat{x}(t, \hat{x}_0, \omega, \mathbf{z}) := \{x \in X \mid \exists \, x_0 \in \hat{x}_0 \text{ and } \delta \in S(\Delta) \text{ s.t. } \phi(t, x_0, \varphi(\omega, \delta)) = x \text{ and}$$
$$\phi(\tau, x_0, \varphi(\omega, \delta)) \in h(\mathbf{z}(\tau)) \, \forall \, \tau \in [0, t]\}.$$

The set $\hat{x}(t, \hat{x}_0, \omega, \mathbf{z})$ is called the information state [33] and we will denote it by $\hat{x}(t)$ when $\hat{x}_0$, ω and $\mathbf{z}$ are clear from the context. We note that if the set of initial conditions $\hat{x}_0$ is compact, then the information state $\hat{x}(t, \hat{x}_0, \omega, \mathbf{z})$ is compact by the compactness of the set-valued flow and the closed value property of the output map $h(z)$.

Problem 1. *(Dynamic Feedback Safety Control Problem)* Given a game structure $\mathscr{G}$, determine the set

$$\bar{\mathcal{W}} := \left\{ A \in 2^X \mid \exists \, \omega \in S(\Omega) \text{ s.t. } \forall \, \mathbf{z} \in S(O) \text{ and } \forall \, t \in \mathbb{R}_+ \text{ we have } \hat{x}(t, A, \omega, \mathbf{z}) \cap \mathbf{B} = \emptyset \right\}$$

and a set-valued map $G : 2^X \rightrightarrows \Omega$ such that for initial convex sets $A \in \bar{\mathcal{W}}$, we have $\hat{x}(t, A, \omega, \mathbf{z}) \cap \mathbf{B} = \emptyset$ for all $t \in \mathbb{R}_+$ and $\mathbf{z} \in S(O)$ when $\omega(\tau) \in G(\hat{x}(\tau, A, \omega, \mathbf{z}))$ for all $\tau \in \mathbb{R}_+$.

This problem can be interpreted as one of determining the set of all initial state uncertainties $A \in 2^X$ for which a control map exists, that on the basis of the measurable signals, guarantees that the information state never intersects $\mathbf{B}$.

Problem 2. *(Static Feedback Safety Control Problem)* Given a game structure $\mathscr{G}$ with $O = X$ and h the identity map, determine the set

$$\mathcal{W} := \{x \in X \mid \exists \, \omega \in S(\Omega) \text{ s.t. } \forall \, \delta \in S(\Delta) \text{ and } \forall \, t \in \mathbb{R}_+ \text{ we have } \phi(t, x, \varphi(\omega, \delta)) \notin \mathbf{B}\}$$

and a set-valued map $g : X \rightrightarrows \Omega$ such that for initial conditions $x \in \mathcal{W}$, we have that $\phi(t, x, \varphi(\omega, \delta)) \notin \mathbf{B}$ for all $\delta \in S(\Delta)$ and $t \in \mathbb{R}_+$ when $\omega(\tau) \in g(\phi(\tau, x, \varphi(\omega, \delta)))$ for all $\tau \in \mathbb{R}_+$.

This problem can be interpreted as one of determining the set of all initial states $x \in X$ for which a static feedback map exists such that the flow of the system never enters $\mathbf{B}$ for all possible disturbance signals δ.

5 Problem Solution

In this section, we propose the solution to Problems 1 and 2 by first computing the complement to the sets $\bar{\mathcal{W}}$ and $\mathcal{W}$, then explicitly computing a dynamic and a static feedback map.

5.1 Computation of the Sets $\bar{\mathcal{W}}$ and $\mathcal{W}$

Consider $C := X \backslash \mathcal{W}$. This set is named the *capture set* as it represents the set of all initial states for which no matter what control is applied, there is a disturbance that drives the flow into **B**. It is mathematically represented as

$$C = \{x \in X \mid \forall\, \omega \in S(\Omega), \exists\, \delta \in S(\Delta) \text{ and } t \in \mathbb{R}_+ \text{ s.t. } \phi(t, x, \varphi(\omega, \delta)) \in \mathbf{B}\}.$$

For a fixed control signal $\bar{\omega} \in S(\Omega)$, we define the *restricted capture set* $C_{\bar{\omega}}$ to be the capture set when the control signal is fixed to $\bar{\omega}$. Mathematically, it is expressed as

$$C_{\bar{\omega}} = \{x \in X \mid \exists\, \delta \in S(\Delta) \text{ and } t \in \mathbb{R}_+ \text{ s.t. } \phi(t, x, \varphi(\bar{\omega}, \delta)) \in \mathbf{B}\}.$$

The restricted capture sets form the basis for our solution to Problems 1 and 2. In the simple example presented in Section 1, two restricted capture sets of relevance, $C_{\omega_{\mathcal{H}}}$ and $C_{\omega_{\mathcal{L}}}$, are represented in Figure 1. More generally, for an order preserving game structure define the constant control $\omega_{\mathcal{L}} := (\omega_H^1, \omega_L^2)$ and $\omega_{\mathcal{H}} := (\omega_L^1, \omega_H^2)$, and corresponding control signals $\omega_{\mathcal{L}}(t) := \omega_{\mathcal{L}}$ and $\omega_{\mathcal{H}}(t) := \omega_{\mathcal{H}}$ for all $t \in \mathbb{R}_+$. For all $\omega \in S(\Omega)$, we have that

$$\omega_{\mathcal{L}} \trianglelefteq \omega \trianglelefteq \omega_{\mathcal{H}}. \tag{1}$$

Similarly, define the constant disturbance $\delta_{\mathcal{L}} := (\delta_H^1, \delta_L^2)$ and $\delta_{\mathcal{H}} := (\delta_L^1, \delta_H^2)$, and corresponding disturbance signals $\delta_{\mathcal{L}}(t) := \delta_{\mathcal{L}}$ and $\delta_{\mathcal{H}}(t) := \delta_{\mathcal{H}}$ for all $t \in \mathbb{R}_+$. For all $\delta \in S(\Delta)$, we have that

$$\delta_{\mathcal{L}} \trianglelefteq \delta \trianglelefteq \delta_{\mathcal{H}}. \tag{2}$$

We now state the main results of this paper.

Lemma 1. *Consider order preserving game structure* $\mathcal{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$ *with a convex set* $A \subset X$. *Let* $\omega \in S(\Omega)$ *and* $\gamma \in C^0(I, \mathbb{R}^2)$ *o.p.c. with* $\inf \tau_1(A) < \max \tau_1(\gamma(I))$. *Then,* $\gamma(I) \cap \bigcup_{\delta \in S(\Delta)} \phi_1(t, A, \varphi(\omega, \delta)) = \emptyset$ *for all* $t \in \mathbb{R}_+$ *if and only if* $\phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta_{\mathcal{L}})) > \gamma(I)$ *or* $\phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta_{\mathcal{H}})) \prec \gamma(I)$.

Lemma 1 states that the flow in the (x_1^1, x_1^2) subspace generated from the convex set of initial conditions A and control ω can avoid an o.p.c. path $\gamma(I)$ for all disturbance signals if and only if the disturbance signal $\delta_{\mathcal{L}}$ takes the trajectory of ϕ_1 above $\gamma(I)$ or the disturbance $\delta_{\mathcal{H}}$ takes the trajectory of ϕ_1 below $\gamma(I)$. This result can be generalized to connected sets $A \subset X$ such that $\tau_{1,2}(A)$ is convex, that is, to cases where *only* the projection of the set A onto the subspace X_1 need be convex.

Theorem 1. *Consider order preserving game structure* $\mathcal{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$ *with a convex set* $A \subset X$. *Then, the following statements are equivalent*

   (i) $A \cap C_{\omega_{\mathcal{L}}} \neq \emptyset$ *and* $A \cap C_{\omega_{\mathcal{H}}} \neq \emptyset$;

   (ii) For all $\omega \in S(\Omega)$, *there exist* $\delta \in S(\Delta)$ *and* $t \in \mathbb{R}_+$ *such that* $\phi(t, A, \varphi(\omega, \delta)) \cap \mathbf{B} \neq \emptyset$.

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Proof. ($\Leftarrow$ Contrapositive) By the definition of the restricted capture set, we have that if $A \cap C_{\omega_{\mathcal{L}}} = \emptyset$ then $\phi(t, A, \varphi(\omega_{\mathcal{L}}, \delta)) \cap \mathbf{B} = \emptyset$ for all $t \in \mathbb{R}_+$ and $\delta \in S(\Delta)$. Similarly, if $A \cap C_{\omega_{\mathcal{H}}} = \emptyset$ then $\phi(t, A, \varphi(\omega_{\mathcal{H}}, \delta)) \cap \mathbf{B} = \emptyset$ for all $t \in \mathbb{R}_+$ and $\delta \in S(\Delta)$.

($\Rightarrow$ Construction) Consider an arbitrary $\omega \in S(\Omega)$. Since $A \cap C_{\omega_{\mathcal{L}}} \neq \emptyset$ and $A \cap C_{\omega_{\mathcal{H}}} \neq \emptyset$, the definition of the restricted capture set implies that there are $x, y \in A, \delta_1, \delta_2 \in S(\Delta)$ and $t_1, t_2 \in \mathbb{R}_+$ such that $\phi(t_1, x, \varphi(\omega_{\mathcal{L}}, \delta_1)) \in \mathbf{B}$ and $\phi(t_2, y, \varphi(\omega_{\mathcal{H}}, \delta_2)) \in \mathbf{B}$. Let $\nu, \kappa \in \mathbb{R}^2$ be such that $\nu = \phi_1(t_1, x, \varphi(\omega_{\mathcal{L}}, \delta_1))$ and $\kappa = \phi_1(t_2, y, \varphi(\omega_{\mathcal{H}}, \delta_2))$. Since $\kappa, \nu \in B$ and B is an o.p.c. set, there exists an o.p.c. path $\gamma \in C^0(I, B)$ with $\gamma(0) = \kappa$ and $\gamma(1) = \nu$.

From equations (1)-(2) and the order preserving property of φ with respect to control ω and disturbance δ, we have that $\varphi(\omega_{\mathcal{L}}, \delta_1) \trianglelefteq \varphi(\omega, \delta_{\mathcal{H}})$. From Proposition 1, we have that $\phi_1(\mathbb{R}_+, x, \varphi(\omega_{\mathcal{L}}, \delta_1)) \preceq \phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{H}}))$. Since $\phi_1(t_1, x, \varphi(\omega_{\mathcal{L}}, \delta_1)) = \nu \in \gamma(I)$ and $x \in A$, this in turn implies that

$$\phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta_{\mathcal{H}})) \not\prec \gamma(I). \tag{3}$$

From equations (1)-(2) and the order preserving property of φ with respect to control ω and disturbance δ, we have that $\varphi(\omega, \delta_{\mathcal{L}}) \trianglelefteq \varphi(\omega_{\mathcal{H}}, \delta_2)$. From Proposition 1, we have that $\phi_1(\mathbb{R}_+, y, \varphi(\omega, \delta_{\mathcal{L}})) \preceq \phi_1(\mathbb{R}_+, y, \varphi(\omega_{\mathcal{H}}, \delta_2))$. Since also $\phi_1(t_2, y, \varphi(\omega_{\mathcal{H}}, \delta_2)) = \kappa \in \gamma(I)$ and $y \in A$, we have that

$$\phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta_{\mathcal{L}})) \not\succ \gamma(I). \tag{4}$$

Note that $y_1 < \kappa_1$ from condition (iii) of Definition 4, implying that $\inf \tau_1(A) < \max \tau_1(\gamma(I))$. Therefore, equations (3)-(4) and Lemma 1 imply that $\gamma(I) \cap \bigcup_{\delta \in S(\Delta)} \phi_1(t, A, \varphi(\omega, \delta)) \neq \emptyset$ for some $t \in \mathbb{R}_+$. This in turn implies, since $\gamma(I) \subset B$, that there are $\bar{\delta} \in S(\Delta)$ and $t \in \mathbb{R}_+$ such that $\phi_1(t, A, \varphi(\omega, \bar{\delta})) \cap B \neq \emptyset$. This leads to $\phi(t, A, \varphi(\omega, \bar{\delta})) \cap \mathbf{B} \neq \emptyset$. Since this holds for arbitrary $\omega \in S(\Omega)$, we have completed the proof. $\square$

Corollary 1. *For an order preserving game structure* $\mathscr{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$, *we have that* $C = C_{\omega_{\mathcal{H}}} \cap C_{\omega_{\mathcal{L}}}$.

Proof. ($\subset$) This follows from the definition of C. ($\supset$) Suppose we have that the initial condition $x \in C_{\omega_{\mathcal{H}}} \cap C_{\omega_{\mathcal{L}}}$. Consider any input signal $\omega \in S(\Omega)$. Since $\tau_{1,2}(\{x\})$ is trivially convex, by Theorem 1 there are $\delta \in S(\Delta)$ and $t \in \mathbb{R}_+$ such that $\phi(t, \{x\}, \varphi(\omega, \delta)) \cap \mathbf{B} \neq \emptyset$, implying $x \in C$. $\square$

Theorem 1 states that an initial convex state uncertainty is taken to intersect $\mathbf{B}$ independently of the control input if and only if it intersects both restricted capture sets $C_{\omega_{\mathcal{H}}}$ and $C_{\omega_{\mathcal{L}}}$. By the corollary, a known initial state is taken to $\mathbf{B}$ independently of the control input if and only if it is in both $C_{\omega_{\mathcal{H}}}$ and $C_{\omega_{\mathcal{L}}}$.

5.2 The Control Map

For an order preserving game structure $\mathscr{G}$, if an initial convex state uncertainty A does not intersect both $C_{\omega_{\mathcal{H}}}$ and $C_{\omega_{\mathcal{L}}}$, from Theorem 1 a control ω exists such that $\phi(t, A, \varphi(\omega, \delta))$ never intersects $\mathbf{B}$ for all δ. Since $\hat{x}(t, A, \omega, \mathbf{z}) \subseteq \bigcup_{\delta \in S(\Delta)} \phi(t, A, \varphi(\omega, \delta))$, there must also exist a control ω such that $\hat{x}(t, A, \omega, \mathbf{z})$ never intersects $\mathbf{B}$. We thus construct such a control as a feedback map from the current state uncertainty $\hat{x}$. For this purpose, define for an element $Z \in 2^X$, the set-valued map $G : 2^X \rightrightarrows \Omega$ as

$$G(Z) := \begin{cases} \omega_{\mathcal{L}} & \text{if } Z \cap C_{\omega_{\mathcal{H}}} \neq \emptyset \text{ and } Z \cap \partial C_{\omega_{\mathcal{L}}} \neq \emptyset \text{ and } Z \cap C_{\omega_{\mathcal{L}}} = \emptyset \\ \omega_{\mathcal{H}} & \text{if } Z \cap C_{\omega_{\mathcal{L}}} \neq \emptyset \text{ and } Z \cap \partial C_{\omega_{\mathcal{H}}} \neq \emptyset \text{ and } Z \cap C_{\omega_{\mathcal{H}}} = \emptyset \\ \omega_{\mathcal{L}} & \text{if } Z \cap \partial C_{\omega_{\mathcal{H}}} \neq \emptyset, Z \cap \partial C_{\omega_{\mathcal{L}}} \neq \emptyset \text{ and } Z \cap (C_{\omega_{\mathcal{H}}} \cup C_{\omega_{\mathcal{L}}}) = \emptyset \\ \Omega & \text{otherwise.} \end{cases} \tag{5}$$

We call the pair $(\mathscr{G}, G)$ a *control system*, where given the initial conditions $A \subset X$ and measurement $\mathbf{z} \in S(O)$, the control system $(\mathscr{G}, G)$ generates the *feedback* $\boldsymbol{\omega}^{cl} \in S(\Omega)$ and the *closed-loop* information state $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z})$. The feedback must satisfy the set-valued map G for all time, namely $\boldsymbol{\omega}^{cl}(t) \in G(\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}))$ for all $t \in \mathbb{R}$.

We next show that the control system $(\mathscr{G}, G)$, where $\mathscr{G}$ is an order preserving game structure and G is given by (5), generates a closed-loop information state that never intersects $\mathbf{B}$ provided the initial conditions $A \subset X$ are compact, connected, and $A \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset$ or $A \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$.

Theorem 2. *Let $\mathscr{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$ be an order preserving game structure, $(\mathscr{G}, G)$ be the control system generated by the static set-valued feedback (5), and let $A \subset X$ be compact and convex. If $A \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset$ or $A \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$, then for arbitrary $\mathbf{z} \in S(O)$ we have that $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap \mathbf{B} = \emptyset$ for all $t \in \mathbb{R}_+$ under $(\mathscr{G}, G)$.*

Proof. First, note that if $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}} = \emptyset$ for some $\omega \in S(\Omega)$, then necessarily $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap \mathbf{B} = \emptyset$ because $\mathbf{B} \subset C_{\boldsymbol{\omega}}$. Thus, we show that if $A \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset$ or $A \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$, then $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset$ or $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$ for all $t \in \mathbb{R}_+$.

We proceed by constructing a modified control system $(\mathscr{G}, \hat{G})$ with a dynamic set-valued map $\hat{G}$, that differs from G only if the argument $Z \subset X$ is such that $Z \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} \neq \emptyset$ and $Z \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} \neq \emptyset$. Denote the closed-loop information state generated by the modified control system as $\hat{y}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z})$. We will show that $\hat{y}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$ or $\hat{y}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset$ for all $t \in \mathbb{R}_+$. We then show that this implies that the feedback generated by the modified control system $(\mathscr{G}, \hat{G})$ is no different from the feedback generated by the original control system $(\mathscr{G}, G)$. Thus, we also have that $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$ or $\hat{x}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset$ for all $t \in \mathbb{R}_+$.

We now define the dynamic set-valued feedback $\hat{G} : \mathbb{R}_+ \times S(2^X) \rightrightarrows \Omega$ as follows. For the time varying set $\mathbf{Z} \subset S(2^X)$ and time $t \in \mathbb{R}_+$, we define $\hat{G}(t, \mathbf{Z})$ as

$$\hat{G}(t, \mathbf{Z}) := \begin{cases} G(\mathbf{Z}(t)) & \text{if } \mathbf{Z}(t) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset \text{ or } \mathbf{Z}(t) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset \\ G(\mathbf{Z}(t^*)) & \text{else, where } t^* := \sup\{\zeta \in [0, t] \mid \mathbf{Z}(\zeta) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset \text{ or } \mathbf{Z}(\zeta) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} = \emptyset\}. \end{cases} \tag{6}$$

We will now show that the closed-loop information state $\hat{y}^{cl}(t, A, \boldsymbol{\omega}^{cl}, \mathbf{z})$ generated by the control system $(\mathscr{G}, \hat{G})$ never intersects both $C_{\boldsymbol{\omega}_{\mathcal{H}}}$ and $C_{\boldsymbol{\omega}_{\mathcal{L}}}$ at a single time $t \in \mathbb{R}$.

We proceed by contradiction. Suppose that given the measurement $\mathbf{z} \in S(O)$, there exists a time $t_1 > 0$ and feedback $\bar{\omega}^{cl} \in S(\Omega)$ generated by $(\mathscr{G}, \hat{G})$ such that $\hat{y}^{cl}(t_1, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} \neq \emptyset$ and $\hat{y}^{cl}(t_1, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} \neq \emptyset$. Define the times

$$t_{\mathcal{L}} := \inf\{t \in [0, t_1] \mid \hat{y}^{cl}(\zeta, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} \neq \emptyset \; \forall \; \zeta \in [t, t_1]\} \tag{7}$$

$$t_{\mathcal{H}} := \inf\{t \in [0, t_1] \mid \hat{y}^{cl}(\zeta, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} \neq \emptyset \; \forall \; \zeta \in [t, t_1]\}. \tag{8}$$

Let the maximum of these two times be $\bar{t} := \max\{t_{\mathcal{L}}, t_{\mathcal{H}}\}$. We must have one of the following cases: (I) $t_{\mathcal{L}} > t_{\mathcal{H}}$; (II) $t_{\mathcal{L}} < t_{\mathcal{H}}$; (III) $t_{\mathcal{L}} = t_{\mathcal{H}}$.

Case(I). From definition (8), $t_{\mathcal{H}} < \bar{t}$ implies that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{H}}} \neq \emptyset$. We first show that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} = \emptyset$.

Suppose that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\boldsymbol{\omega}_{\mathcal{L}}} \neq \emptyset$. By the definition of the closed-loop information state, there exists $x_0 \in A$ and a disturbance $\boldsymbol{\delta} \in S(\Delta)$ such that $\phi(\bar{t}, x_0, \varphi(\bar{\omega}^{cl}, \boldsymbol{\delta})) \in C_{\boldsymbol{\omega}_{\mathcal{L}}}$ and $\phi(\tau, x_0, \varphi(\bar{\omega}^{cl}, \boldsymbol{\delta})) \in \hat{y}^{cl}(\tau, A, \bar{\omega}^{cl}, \mathbf{z})$ for all $\tau \in [0, \bar{t}]$. For notation, let $\nu := \phi(\bar{t}, x_0, \varphi(\bar{\omega}^{cl}, \boldsymbol{\delta}))$. Since the flow is continuous with respect to initial conditions, one can show that $\mathbf{B}$ open implies that $C_{\boldsymbol{\omega}_{\mathcal{L}}}$ is open. Therefore, we can find $\epsilon > 0$ such that $B(\nu, \epsilon) \subset C_{\boldsymbol{\omega}_{\mathcal{L}}}$. By the continuity of the flow with respect to time, we

can find $\eta > 0$ such that if $t \in]\bar{t} - \eta, \bar{t}]$, then $\phi(t, x_0, \varphi(\bar{\omega}^{cl}, \delta)) \in B(\nu, \epsilon) \subset C_{\omega_\mathcal{L}}$. This implies that $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}} \neq \emptyset$ for all $t \in]\bar{t} - \eta, \bar{t}]$, thus contradicting $\bar{t} = t_\mathcal{L}$ as the infimum in (7).

We next show that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap \partial C_{\omega_\mathcal{L}} \neq \emptyset$. Suppose that instead $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap \partial C_{\omega_\mathcal{L}} = \emptyset$. For notation, let $\hat{y}_0 := \hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z})$. Since A is compact, $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z})$ is compact for all t and $\mathbf{z}$. Now consider the distance $\gamma := d(\partial C_{\omega_\mathcal{L}}, \hat{y}_0)$. If $\gamma = 0$, then the intersection must be non-empty, as both sets are closed. Therefore, we assume that $\gamma > 0$. By the upper hemi-continuity of the set-valued flow, there exists $\eta > 0$ such that for all $t \in [\bar{t}, \bar{t} + \eta[$, we have that $\phi(t, \hat{y}_0, S(U)) \subset B(\hat{y}_0, \gamma/2)$. By the definition of the closed-loop information state, for all $t \geq \bar{t}$ we have that $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \subset \phi(t, \hat{y}_0, S(U))$. This implies that for all $t \in [\bar{t}, \bar{t}+\eta[$ we have $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{H}} = \emptyset$, since $d(\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}), C_{\omega_\mathcal{H}}) > \gamma/2 > 0$. This contradicts $\bar{t} = t_\mathcal{L}$ as given in equation (7), hence we must have that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap \partial C_{\omega_\mathcal{L}} \neq \emptyset$.

We have thus shown that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{H}} \neq \emptyset$, $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap \partial C_{\omega_\mathcal{L}} \neq \emptyset$ and $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}} = \emptyset$. From the definition of the modified dynamic set-valued feedback map $\hat{G}$ given in (6), we must necessarily have that $\bar{\omega}^{cl}(\bar{t}) = \omega_\mathcal{L} = \hat{G}(\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}))$. From definitions (7) and (8), we therefore have that $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{H}} \neq \emptyset$ and $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}} \neq \emptyset$ for all $t \in [\bar{t}, t_1]$. Therefore, by the definition of $\hat{G}$ in equation (6), we have that $\bar{\omega}^{cl}(t) = \omega_\mathcal{L} = \hat{G}(\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}))$ for all $t \in [\bar{t}, t_1]$. Let $v \in \hat{y}^{cl}(t_1, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}}$ and choose $w \in \hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z})$ such that $\phi(t_1 - \bar{t}, w, \varphi(\omega_\mathcal{L}, \delta)) = v$ for some $\delta \in S(\Delta)$ (note that such a w exists by the definition of the information state $\hat{y}$). Since $v \in C_{\omega_\mathcal{L}}$ and $\omega(t) = \omega_\mathcal{L}$ for all $t \in [\bar{t}, t_1]$, we must have that $w \in C_{\omega_\mathcal{L}}$ by the definition of $C_{\omega_\mathcal{L}}$. This leads to a contradiction, since we assumed that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}} = \emptyset$. As a consequence, such a time t_1 for which Case(I) holds cannot exist.

For Case(II), an equivalent argument holds by interchanging $\omega_\mathcal{L}$ with $\omega_\mathcal{H}$, and $C_{\omega_\mathcal{L}}$ with $C_{\omega_\mathcal{H}}$, then showing that this leads to a contradiction of $t_\mathcal{H}$ as defined in (8).

For Case(III), the argument is similar. First, it can be shown that $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap \partial C_{\omega_\mathcal{L}} \neq \emptyset$ and $\hat{y}^{cl}(\bar{t}, A, \bar{\omega}^{cl}, \mathbf{z}) \cap \partial C_{\omega_\mathcal{H}} \neq \emptyset$ by a continuity argument (similar to the one made in Case(I)). The proof proceeds as in Case(I) with the eventual contradiction regarding the definition $C_{\omega_\mathcal{L}}$, and thus contradicting the existence of $t_\mathcal{L}$ and $t_\mathcal{H}$ as defined in (7) and (8) respectively.

Therefore $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{H}} = \emptyset$ or $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}} = \emptyset$ must hold for all $t \in \mathbb{R}_+$ under any control $\bar{\omega}^{cl} \in S(\Omega)$ generated by $(\mathscr{G}, \hat{G})$. From the definition of G in (5), it must be that $G(\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z})) = \hat{G}(\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}))$ for all $t \in \mathbb{R}_+$. This implies that for every closed-loop information state $\hat{x}^{cl}(t, A, \omega^{cl}, \mathbf{z})$ and feedback ω^{cl} generated by the control system $(\mathscr{G}, G)$, there is a corresponding feedback $\bar{\omega}^{cl}$ and closed-loop information state $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z})$ generated by the control system $(\mathscr{G}, \hat{G})$ such that $\bar{\omega}^{cl} = \omega^{cl}$ and $\hat{y}^{cl}(t, A, \bar{\omega}^{cl}, \mathbf{z}) = \hat{x}^{cl}(t, A, \omega^{cl}, \mathbf{z})$. This implies that $\hat{x}^{cl}(t, A, \omega^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{H}} = \emptyset$ or $\hat{x}^{cl}(t, A, \omega^{cl}, \mathbf{z}) \cap C_{\omega_\mathcal{L}} = \emptyset$ for all $t \in \mathbb{R}_+$. Therefore, the closed-loop information state generated by the control system $(\mathscr{G}, G)$ satisfies $\hat{x}^{cl}(t, A, \omega^{cl}, \mathbf{z}) \cap \mathbf{B} = \emptyset$ for all $t \in \mathbb{R}_+$. $\qquad \square$

We can thus summarize the solutions to Problem 1 and Problem 2 in the two following theorems, respectively.

Theorem 3. *(Solution to Problem 1) For an order preserving game structure $\mathscr{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$, a convex set $\hat{x}_0 \subset X$ is in $\bar{\mathcal{W}}$ if and only if $\hat{x}_0 \cap C_{\omega_\mathcal{H}} = \emptyset$ or $\hat{x}_0 \cap C_{\omega_\mathcal{L}} = \emptyset$. Furthermore, if $\hat{x}_0 \in \bar{\mathcal{W}}$ is also compact, then a dynamic feedback map $G : 2^X \rightrightarrows \Omega$ is given by (5).*

Proof. By Theorem 1, there exists a control signal $\omega \in S(\Omega)$ such that $\phi(t, \hat{x}_0, \varphi(\omega, \delta)) \cap \mathbf{B} = \emptyset$ for all $\delta \in S(\Delta)$ and all $t \in \mathbb{R}_+$ if and only if $\hat{x}_0 \cap C_{\omega_\mathcal{H}} = \emptyset$ or $\hat{x}_0 \cap C_{\omega_\mathcal{L}} = \emptyset$. Assuming that $\mathbf{z}$ is the worst-case observation signal, that is, $\mathbf{z}(t) = \bar{z}$ for all $t \in \mathbb{R}_+$, we have that $\hat{x}(t, \hat{x}_0, \omega, \mathbf{z}) = \bigcup_{\delta \in S(\Delta)} \phi(t, \hat{x}_0, \varphi(\omega, \delta))$ for all $t \in \mathbb{R}_+$. Therefore, there is a control signal $\omega \in S(\Omega)$ such that $\hat{x}(t, \hat{x}_0, \omega, \mathbf{z}) \cap \mathbf{B} = \emptyset$ for all $t \in \mathbb{R}_+$ if and only if $\hat{x}_0 \cap C_{\omega_\mathcal{H}} = \emptyset$ or $\hat{x}_0 \cap C_{\omega_\mathcal{L}} = \emptyset$. By the definition of $\bar{\mathcal{W}}$, we thus have that

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$\hat{x}_0 \in \bar{\mathcal{W}}$ if and only if $\hat{x}_0 \cap C\omega_{\mathcal{H}} = \emptyset$ or $\hat{x}_0 \cap C\omega_{\mathcal{L}} = \emptyset$. Since the set of initial conditions $\hat{x}_0$ is compact, Theorem 2 further shows that the feedback map G given by expression (5) maintains $\hat{x}(t, \hat{x}_0, \omega, \mathbf{z})$ with $\omega(\tau) \in G(\hat{x}(\tau, \hat{x}_0, \omega, \mathbf{z}))$ for all $\tau \in \mathbb{R}_+$ not intersecting $\mathbf{B}$ for all $t \in \mathbb{R}_+$. □

Theorem 4. *(Solution to Problem 2) For an order preserving game structure $\mathcal{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$, the set $\mathcal{W}$ of Problem 2 is given by $\mathcal{W} = X \backslash (C\omega_{\mathcal{H}} \cap C\omega_{\mathcal{L}})$. A feedback map $g : X \rightrightarrows \Omega$ is given by*

$$g(x) := \begin{cases} \omega_{\mathcal{H}} & \text{if } x \in C\omega_{\mathcal{L}} \text{ and } x \in \partial C\omega_{\mathcal{H}} \\ \omega_{\mathcal{L}} & \text{if } x \in C\omega_{\mathcal{H}} \text{ and } x \in \partial C\omega_{\mathcal{L}} \\ \omega_{\mathcal{L}} & \text{if } x \in \partial C\omega_{\mathcal{H}} \text{ and } x \in \partial C\omega_{\mathcal{L}} \\ \Omega & \text{otherwise.} \end{cases}$$

Proof. Direct consequence of Corollary 1 and Theorem 2, in which A is a singleton. □

Since the static feedback map g is equal to the dynamic feedback map G once this map is evaluated on the state x, a separation principle holds for the game structure $\mathcal{G}$ between state estimation and control. This implies that the solution of the dynamic feedback problem does not present additional computational difficulties with respect to the solution of the static feedback problem. Specifically, both solutions rely only on the ability to compute the restricted capture sets $C\omega_{\mathcal{L}}$ and $C\omega_{\mathcal{H}}$. These two sets, as opposed to the original sets of interest $\mathcal{W}$ and $\bar{\mathcal{W}}$, can be computed by backward integration with the control input fixed. Furthermore, if the bad set $\mathbf{B}$ satisfies additional geometric assumptions (Section 6), then this computation only requires the disturbance signals $\delta_{\mathcal{L}}$ and $\delta_{\mathcal{H}}$. Therefore, no min/max optimization problem needs to be solved as it is usually performed when directly computing $\mathcal{W}$. In addition to this simplification, the order preserving properties of $\mathcal{G}$ along with additional assumptions allow the construction of discrete-time linear complexity algorithms for the computation of the restricted capture sets $C\omega_{\mathcal{L}}$ and $C\omega_{\mathcal{H}}$. These algorithms are presented in the next section.

6 Algorithms

By virtue of Theorems 3 and 4, the dynamic and static control Problems 1 and 2 can be solved by only computing the sets $C\omega_{\mathcal{H}}$ and $C\omega_{\mathcal{L}}$. For a class of order preserving systems in discrete-time, we introduce an algorithm for computing the restricted capture set $C\omega$. This algorithm has linear complexity with respect to the number of continuous variables.

The restrictions on the game structure $\mathcal{G}$ imposed are:

    Assumption (a) $f^i(x^i, u^i)$ has no dependency on x^i_1;

    Assumption (b)   The bad set $\mathbf{B}$ is given by $\mathbf{B} := \{x \in X \mid (x^1_1, x^2_1) \in B\}$, with $B := \,]L, H[\subset \mathbb{R}^2$.

This structure of $f^i(x^i, u^i)$ is found, for example, in vector fields derived from Newton's laws with no position dependent forces (such as gravity). The bad set $\mathbf{B}$ generated by the open rectangle set B can represent, for example, the set of all collision configurations between two agents evolving on intersecting paths. If B is a more general bounded o.p.c. set, a rectangular over-approximation can be employed.

6.1 Discrete-Time Model

Seeking digital implementation, we illustrate the algorithm in discrete-time. For agent $i \in \{1, 2\}$, denote the state space $\bar{X}^i := X_2^i \times \ldots \times X_n^i$, the corresponding state $\bar{x}^i \in \bar{X}^i$, and the set of discrete-time signals $D : \mathbb{N} \to U^i$ as $D(U^i)$. Define the discretization of the system (employing forward Euler approximation) for agent $i \in \{1, 2\}$ with step size $\Delta T > 0$, input $\mathbf{u}^i \in D(U^i)$ and step $n \in \mathbb{N}$ as

$$x^i[n + 1] = x^i[n] + \Delta T f^i(x^i[n], \mathbf{u}^i[n]).$$

For the index $n \in \mathbb{N}$, initial condition $x^i \in X^i$, and input signal $\mathbf{u}^i \in D(U^i)$, we denote the discrete-time flow $\Phi^i : \mathbb{N} \times X^i \times D(U^i) \to X^i$ as $\Phi^i(n, x^i, \mathbf{u}^i)$, which satisfies

$$\Phi^i(n + 1, x^i, \mathbf{u}^i) = \Phi^i(n, x^i, \mathbf{u}^i) + \Delta T f^i(\Phi^i(n, x^i, \mathbf{u}^i), \mathbf{u}^i[n - 1]) \text{ for all } n \in \mathbb{N}, \tag{9}$$

where $\Phi^i(0, x^i, \mathbf{u}^i) = x^i$. We assume the discrete flow Φ^i is continuous with respect to input $\mathbf{u}^i \in D(U^i)$. Let $\mathbf{z}^i \in D(O)$ be the output measurement. From Definition 4, the output map is given by $h^i(z^i[n]) = [\inf h^i(z^i[n]), \sup h^i(z^i[n])]$. The jth component of the flow is denoted as $\Phi_j^i(n, x^i, \mathbf{u}^i)$

For the parallel composition of two systems $\Sigma = \Sigma^1 \| \Sigma^2$, the discretization and discrete-time flow extend to

$$\begin{aligned} \Delta T f(x[n], \mathbf{u}[n]) &:= (\Delta T f^1(x^1[n], \mathbf{u}^1[n]), \Delta T f^2(x^2[n], \mathbf{u}^2[n])) \\ \Phi(n, x, \mathbf{u}) &:= (\Phi^1(n, x^1, \mathbf{u}^1), \Phi^2(n, x^2, \mathbf{u}^2)). \end{aligned}$$

The game input map, as in Definition 7, easily extends to discrete-time control signals $\omega \in D(\Omega)$ and disturbance signals $\delta \in D(\Delta)$ as $\mathbf{u}[n] = \varphi(\omega[n], \delta[n])$.

From Assumption (a), it follows that for an initial condition $(x_1, \bar{x}) \in X$ and input $\mathbf{u} \in D(U)$, we have that

$$\Phi_1(n, x, \mathbf{u}) = x_1 + \Phi_1(n, (0, \bar{x}), \mathbf{u}) \text{ for all } n \in \mathbb{N}, \tag{10}$$

where the state $(0, \bar{x})$ represents the initial condition x with the state x_1 set to zero. This property implies that the flow projected onto the subspace X_1 has no dependency on the state x_1 other than the initial condition.

6.2 Restricted Capture Set C_ω Computation

The definition of the discrete-time capture set is the same as in continuous time, however now the index $n \in \mathbb{N}$ replaces time $t \in \mathbb{R}_+$, and the discrete signal $\delta \in D(\Delta)$ replaces the continuous signal $\delta \in S(\Delta)$. This is mathematically represented as

$$C_\omega = \{x \in X \mid \exists n \in \mathbb{N}, \exists \delta \in D(\Delta) \text{ s.t. } \Phi(n, x, \varphi(\omega, \delta)) \in \mathbf{B}\}.$$

To compute the restricted capture set, we introduce the sequences $\{L^i(n, x^i, \omega^i)\}, \{H^i(n, x^i, \omega^i)\} \subset X_1^i$ generated with the state $x^i \in X^i$ and constant control input $\omega^i \in D(\Omega^i)$. These sequences are defined as

$$\begin{aligned} L^i(n, x^i, \omega^i) &:= L^i - \Phi_1^i(n, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_H^i)) \\ H^i(n, x^i, \omega^i) &:= H^i - \Phi_1^i(n, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_L^i)). \end{aligned}$$

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We can combine these sequences for $i \in \{1, 2\}$ and define $L(n, x, \omega) := (L^1(n, x^1, \omega^1), L^2(n, x^2, \omega^2))$, $H(n, x, \omega) := (H^1(n, x^1, \omega^1), H^2(n, x^2, \omega^2))$.

The sequence $\{L(n, x, \omega)\}_{n \in \mathbb{N}}$ represents the backward integration of L with state $(0, \bar{x})$, control input ω and constant disturbance input δ_H. The sequence $\{H(n, x, \omega)\}_{n \in \mathbb{N}}$ represents the backward integration of H with state $(0, \bar{x})$, control input ω and constant disturbance input δ_L. We use both these sequences to define a sequence of rectangle sets as $\{]L(n, x, \omega), H(n, x, \omega)[\}_{k \in \mathbb{N}} \subset \mathbb{R}^2$.

We introduce Algorithm 1, which can be used to compute the restricted capture set C_ω, by recursively computing the elements of the sequence $\{]L(n, x, \omega), H(n, x, \omega)[\}_{n \in \mathbb{N}}$. To accommodate the case of state uncertainty (Section 6.3), the input of Algorithm 1 is a *set* $\hat{x} \subset X$ rather than a singleton $x \in X$.

Algorithm 1 $\tilde{C}_\omega = \text{CaptureSetSlice}(\hat{x}, \omega)$

Input: $(\hat{x}, \omega) \in 2^X \times D(\Omega)$

$n = 1$
loop
   Termination met when the sequence $H(n, \inf \hat{x}, \omega)$ is no longer in the set $Cone_+(\inf \hat{x}_1)$.
   if $\inf \hat{x}_1 \leq H(n, \inf \hat{x}, \omega)$ and $\inf \hat{x}_1 \notin]L(n, \sup \hat{x}, \omega), H(n, \inf \hat{x}, \omega)[$ **then**
     $n = n + 1$
   else
     return $\tilde{C}_\omega = \bigcup_{k \leq n}]L(k, \sup \hat{x}, \omega), H(k, \inf \hat{x}, \omega)[$.
   end if
end loop

Output: $\tilde{C}_\omega \subset X_1$.

We can interpret Algorithm 1 as the backward propagation of the rectangle set $]L, H[$ with control signal ω and all disturbances. This, in turn, by the order preserving properties of the discrete-time flow with respect to the input, only requires the upper bound δ_H and the lower bound δ_L. To show *termination* of Algorithm 1, we note that condition (iii) of Definition 4 implies that the sequence $\{H(n, x, \omega)\}_{n \in \mathbb{N}}$ is strictly monotonically decreasing without limit for any $x \in X$ and $\omega \in D(\Omega)$. Therefore, there must be some finite $n \in \mathbb{N}$ such that $\inf \hat{x}_1 \nleq H(n, \inf \hat{x}, \omega)$, implying termination of Algorithm 2.

Claim 1.

$$C_\omega = \left\{ x \in X \mid x_1 \in \tilde{C}_\omega = \text{CaptureSetSlice}(\{x\}, \omega) \right\}.$$

Proof. Denote $S := \left\{ x \in X \mid x_1 \in \tilde{C}_\omega = \text{CaptureSetSlice}(\{x\}, \omega) \right\}$. We show first that $C_\omega \subseteq S$ and then that $C_\omega \supseteq S$.

($\subseteq$) Let $x \in C_\omega$, then by the definition of C_ω we have that there is $\delta \in D(\Delta)$ and $\bar{n} \in \mathbb{N}$ such that $L \leq \Phi_1(\bar{n}, x, \varphi(\omega, \delta)) \leq H$. From equation (10), we have that

$$L - \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta)) \leq x_1 \leq H - \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta)). \tag{11}$$

From the order preserving property of the game input map with respect to the disturbance and by the order preserving property of the discrete-time flow with respect to the input, we have that

$$\Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta_{\mathcal{L}})) \leq \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta)) \leq \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta_{\mathcal{H}})). \tag{12}$$

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Therefore, from expressions (11) and (12), we have that

$$x_1 \le H - \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta)) \quad \le \quad H - \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta_L)) = H(\bar{n}, x, \omega)$$
$$x_1 \ge L - \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta)) \quad \ge \quad L - \Phi_1(\bar{n}, (0, \bar{x}), \varphi(\omega, \delta_H)) = L(\bar{n}, x, \omega),$$

which imply $x \in S$.

($\supseteq$) Let $x \in S$, for agent $i \in \{1, 2\}$ we have that $x_1^i \le H^i(\bar{n}, x^i, \omega^i) = H^i - \Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_L^i))$ and $x_1^i \ge L^i(\bar{n}, x^i, \omega^i) = L^i - \Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_H^i))$ for some $\bar{n} \in \mathbb{N}$. We can rearrange these inequalities to give $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_L^i)) \le H^i - x_1^i$ and $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_H^i)) \ge L^i - x_1^i$. If either $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_L^i)) \ge L^i - x_1^i$ or $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_H^i)) \le H^i - x_1^i$, we have that there is a disturbance δ such that $x_1^i + \Phi_1^i(\bar{n}, (0, \bar{x}), \varphi^i(\omega^i, \delta^i))) = \Phi_1^i(\bar{n}, x^i, \varphi(\omega, \delta)) \in]L^i, H^i[$. If neither of these two cases is satisfied, the following inequalities are satisfied: $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_L^i)) < L - x_1$ and $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta_H^i)) > H - x_1$. Since $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \cdot)) : D(\Delta^i) \to X_1^i$ is a continuous function and $D(\Delta^i)$ is a connected metric space with $\Delta^i = [\delta_L^i, \delta_H^i]$, by the intermediate value theorem there must be $\delta^i \in D(\Delta^i)$ such that $\Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta^i)) = w \in]L^i - x_1^i, H^i - x_1^i[$. As a consequence, for such a δ^i we have that $x_1^i + \Phi_1^i(\bar{n}, (0, \bar{x}^i), \varphi^i(\omega^i, \delta^i)) = \Phi_1^i(\bar{n}, x^i, \varphi^i(\omega^i, \delta^i)) \in]L^i, H^i[$. Since this holds for arbitrary $i \in \{1, 2\}$, we have shown that $x \in C_\omega$. $\qquad\qquad\square$

Note that the sets C_ω are $2n$ dimensional. Claim 1 shows that these high dimensional sets can be computed by just computing a sequence of lower $\{L(n, x, \omega)\}_{n \in \mathbb{N}}$ and upper $\{H(n, x, \omega)\}_{n \in \mathbb{N}}$ bounds in X_1, which are parameterized by the $2n$ state variables x. For any fixed value of $x \in X$, the union of intervals $\cup_{n \in \mathbb{N}}]L(n, x, \omega), H(n, x, \omega)[$ over all $n \in \mathbb{N}$ represents the two dimensional slice of C_ω corresponding to the state x.

The boundary of the capture set ∂C_ω must be reinterpreted, as now the discrete-time flow can enter the interior of the capture set without touching the boundary. We provide a definition of the capture set boundary ∂C_ω as

$$\partial C_\omega := \{x \in X \backslash C_\omega \mid \exists \, \delta \in \Delta \text{ s.t. } x + \Delta T f(x, \varphi(\omega, \delta)) \in C_\omega\}. \tag{13}$$

According to this definition, a state outside of the restricted capture set is said to be on the boundary of the restricted capture set, if there is some disturbance such that the state is mapped inside the capture set in one step.

6.3 Dynamic Feedback Implementation

Since the dynamics of the system are order preserving with respect to the state and to the input, we construct a state estimator that keeps track of only the lower and upper bounds of the information state similar to the estimator proposed in [23]. Let $\vee \hat{x} := \sup \hat{x}$ and $\wedge \hat{x} := \inf \hat{x}$ denote the upper and lower bounds, respectively, of the set of possible current states $\hat{x}$ (the sup and inf are taken component-wise in accordance to the partial ordering defined on $(X, \le)$). Then, a state estimate $\hat{x}[n]$ is constructed with Algorithm 2, by only updating the upper and lower bounds of $\hat{x}[n-1]$. To construct the state estimate, first the previous state estimate is mapped forward under the discrete update map with the control input supplied and all possible disturbances. Then, the measurement is used to further restrict the set of all possible compatible states. Conditions leading to estimator convergence are provided in [23] for a class of systems.

Algorithm 2 $\hat{x}[n] = \text{StateEstimate}(\hat{x}[n-1], \omega[n-1], z[n])$

Input: $(\hat{x}[n-1], z[n]) \in 2^X \times O$

Update state estimate.
$\vee\hat{x}[n] = \inf\{\Delta T f(\vee\hat{x}[n-1], \varphi(\omega[n-1], \delta_H)), \sup h(z[n])\}.$
$\wedge\hat{x}[n] = \sup\{\Delta T f(\wedge\hat{x}[n-1], \varphi(\omega[n-1], \delta_L)), \inf h(z[n])\}.$

Return state estimate with upper and lower bounds.
return $\hat{x}[n] = [\wedge\hat{x}[n], \vee\hat{x}[n]].$

Output: $\hat{x}[n] \subset X$.

To implement the closed-loop feedback $G : 2^X \rightrightarrows U$ given by equation (5) from Section 5.2, one must check whether the state estimate $\hat{x}[n]$ intersects C_{ω_H} and C_{ω_L}. Since the sequence $L(k, x, \omega)$ is order reversing in the argument x, a sufficient condition guaranteeing that $\hat{x}[n] \cap C_{\omega} = \emptyset$ is that

$$\hat{x}_1[n] \cap \bigcup_{k \in \mathbb{N}}]L(k, \vee\hat{x}[n], \omega), H(k, \wedge\hat{x}[n], \omega)[= \emptyset. \tag{14}$$

We introduce Algorithm 3, which can be used to compute the feedback $\omega[n]$ generated by the set-valued map G by using the current state $\hat{x}[n]$ and the state prediction $\hat{x}[n + 1]$.

We can interpret Algorithm 3 as the discrete-time implementation of the set-valued map G, as defined in (5). The algorithm is comprised of a series of steps. First, capture set slices are constructed with Algorithm 1 for the state prediction. If the state prediction $\hat{x}[n + 1]$ has non-empty intersection with each restricted capture set, as established by equation (14), then the state estimate $\hat{x}[n]$ either has non-empty intersection or is on the boundary of each restricted capture set. The state estimate $\hat{x}[n]$ is on the boundary of a restricted capture set, as defined in (13), if the state estimate $\hat{x}[n]$ has empty intersection with the corresponding capture set slice constructed with Algorithm 1. If the intersection is non-empty, then the state estimate $\hat{x}[n]$ has non-empty intersection with the restricted capture set. Lastly, control is evaluated with the set-valued map G based on the restricted capture set membership established.

The closed-loop control system is implemented with Algorithm 4, where the feedback and state estimate are given by $(\omega[n], \hat{x}[n]) = \text{ControlSystem}(\hat{x}[n-1], \mathbf{z}[n])$. We can summarize Algorithm 4 as follows. First, the state estimate is constructed with Algorithm 2. Next, a state prediction is constructed by mapping the current state estimate forward with all possible disturbance signals. Finally, control is evaluated with Algorithm 3 based on current state estate estimate and state prediction.

7 Simulation and Experimental Results

In this section, we illustrate the application of the algorithms outlined in Section 6 to the two-vehicle collision avoidance problem introduced in Section 1, in which we now consider disturbances, imperfect state information, and higher order piecewise continuous vehicle dynamics.

In-vehicle cooperative active safety and related technologies continue to be examined world-wide by government and industry consortium, such as the Crash Avoidance Metrics Partnership (CAMP)

Algorithm 3 $\omega = \text{FeedbackMap}(\hat{x}[n+1], \hat{x}[n])$

Input: $(\hat{x}[n+1], \hat{x}[n]) \in 2^X \times 2^X$

Construct capture set slices for state prediction.
$\tilde{C}_{\omega_L} = \text{CaptureSetSlice}(\hat{x}[n+1], \omega_L)$, $\tilde{C}_{\omega_H} = \text{CaptureSetSlice}(\hat{x}[n+1], \omega_H)$

Check if predicted state $\hat{x}[n+1]$ intersects both capture set slices.
if $\hat{x}[n+1] \cap \tilde{C}_{\omega_L} \neq \emptyset$ a*nd* $\hat{x}[n+1] \cap \tilde{C}_{\omega_H} \neq \emptyset$ **then**

   Construct capture set slices for current state.
   $\tilde{C}_{\omega_L} = \text{CaptureSetSlice}(\hat{x}[n], \omega_L)$, $\tilde{C}_{\omega_H} = \text{CaptureSetSlice}(\hat{x}[j], \omega_H)$

   Determine control according to equation (5).
   if $\hat{x}_1[n] \cap \tilde{C}_{\omega_L} = \emptyset$ a*nd* $\hat{x}_1[n] \cap \tilde{C}_{\omega_H} \neq \emptyset$ **then**
      $\omega = \omega_L$
   else if $\hat{x}_1[n] \cap \tilde{C}_{\omega_L} \neq \emptyset$ a*nd* $\hat{x}_1[n] \cap \tilde{C}_{\omega_H} = \emptyset$ **then**
      $\omega = \omega_H$
   else
      $\omega = \omega_L$
   end if

else
   No control specified.
   $\omega \in \Omega$
end if

Output: $\omega \subset \Omega$.

Algorithm 4 $(\omega^{cl}[n], \hat{x}[n]) = \text{ControlSystem}(\hat{x}[n-1], \mathbf{z}[n])$

Input: $(\hat{x}[n-1], \mathbf{z}[n]) \in 2^X \times O$

Update state estimate.
$\hat{x}[n] = \text{StateEstimate}(\hat{x}[n-1], \mathbf{z}[n])$

Construct state prediction.
$\hat{x}[n+1] = [\Delta T f(\vee \hat{x}[n], \varphi(\omega[n], \delta_L)), \Delta T f(\wedge \hat{x}[n], \varphi(\omega[n], \delta_H))]$

Compute closed-loop feedback.
$\omega^{cl}[n] = \text{FeedbackMap}(\hat{x}[n+1], \hat{x}[n])$

Output: $(\omega^{cl}[n], \hat{x}[n]) \in \Omega \times 2^X$

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[2], the Vehicle Infrastructure Integration Consortium (VIIC) [3, 4] in the U.S., the Car2Car Commu-
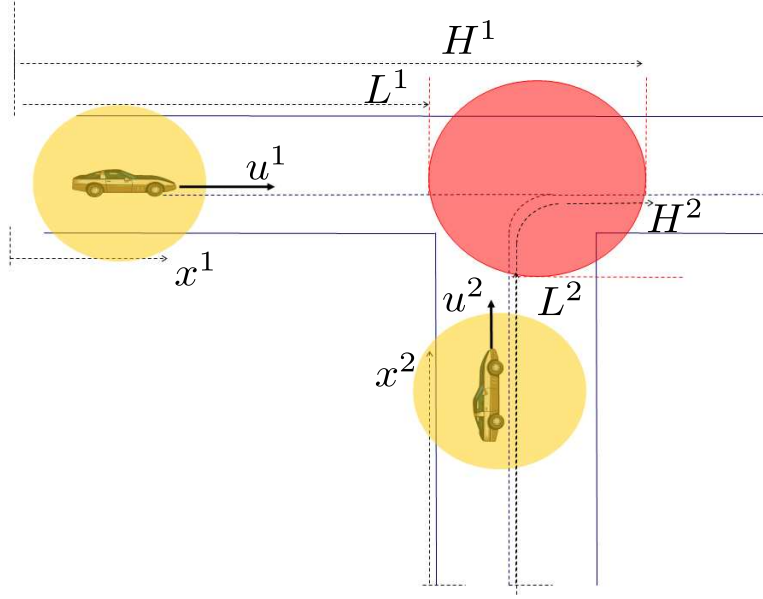
Figure 3: Vehicles approaching a "T" intersection. A collision occurs if two vehicles are in the set B at the same time.

nications Consortium in Europe [1], the Advanced Safety Vehicle project 3 (ASV3) in Japan, and by university research centers such as the Virginia Tech Transportation Institute (VTTI) and the California PATH. In the near future, ITS is expected to become more comprehensive by connecting vehicles with each other and with the surrounding road infrastructure through vehicle-to-vehicle (V2V) and vehicle-to-infrastructure (V2I) wireless communication.

Here, we consider three different scenarios. In the first scenario, the cooperative case, we assume V2V communication. The two vehicles thus share information and cooperate to prevent a potential collision. In the second scenario, the competitive case, we assume that the two vehicles cannot communicate with each other, for example, only one of the two vehicles is equipped with the on-board active safety system. This scenario is of high interest, as any realistic deployment of cooperative active safety systems will not be universally equipped on all vehicles. The third scenario assumes V2V communication and thus cooperation between the two vehicles. However, we assume that the dynamic model of the vehicles is subject to modeling uncertainty. For this combined case, experimental results on a concrete in-lab implementation are presented. In all of these three cases, we consider the traffic intersection instance depicted in Figure 3 as a reference.

The longitudinal dynamics of each vehicle along its path can be modeled employing Newton's laws. Let $p \in \mathbb{R}$ denote the longitudinal displacement along the vehicle path. The longitudinal vehicle dynamics can thus be written as

$$\ddot{p} = [\mathcal{R}^2/(J_w + \mathcal{M}\mathcal{R}^2)](f_w - f_{brake} - \frac{\rho_{air}}{2}C_D A_f v^2 - C_{rr}\mathcal{M}g - \mathcal{M}g\sin(\theta_{road})),$$

in which $\mathcal{R}$ is the tire radius, J_w is the wheel inertia, $\mathcal{M}$ is the mass of the vehicle, $f_w = \tau_w\mathcal{R}$ where τ_w is the drive shaft output torque, f_{brake} is the brake force, ρ_{air} is the air density, C_D is the drag coefficient, A_f is the projected front area of the vehicle, v is the longitudinal vehicle velocity, C_{rr} is the rolling resistance coefficient, g is the gravity constant, and θ_{road} is the road gradient. For more details on this model, the reader is referred to [48] and to the references therein. For automatic driving, f_w and f_{brake}

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are control inputs to the longitudinal dynamics of the vehicle. Assuming that the road is flat and that the air drag term is negligible, we can re-write the longitudinal dynamics as

$$\ddot{p} = a\, u + b, \tag{15}$$

in which $u = f_w - f_{brake}$ is the total force, which is the control input to the vehicle, $a = \mathcal{R}^2/(J_w + \mathcal{M}\mathcal{R}^2)$, and $b = -\mathcal{R}^2/(J_w + \mathcal{M}\mathcal{R}^2)\, C_{rr}\mathcal{M}g$.

For vehicle $i \in \{1, 2\}$, we denote (see Figure 3) the longitudinal displacement along its path by x_1^i and the longitudinal speed by x_2^i. As a consequence, the longitudinal dynamics for vehicle $i \in \{1, 2\}$ can be re-written as

$$\begin{aligned} \dot{x}_1^i &= x_2^i \\ \dot{x}_2^i &= a^i u^i + b^i. \end{aligned}$$

In order to prevent the vehicle from stopping (to prevent the trivial solution in which the vehicles come to a stop) and from exceeding a maximum speed (to respect road speed limitations), we consider the hybrid system depicted in Figure 4. For each vehicle subsystem Σ^i, we choose for $z^i \in \mathbb{R}^2$ an output
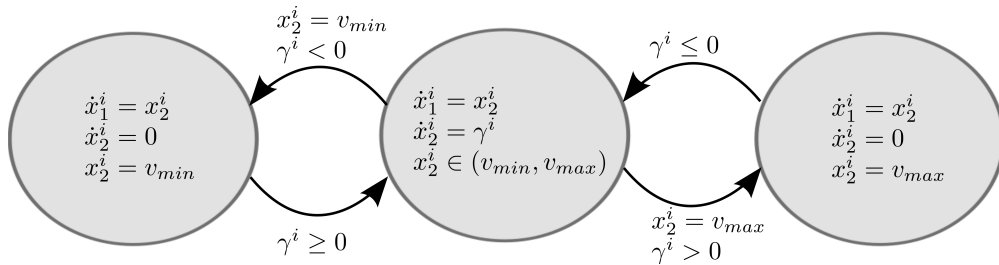

Figure 4: Hybrid system modeling the vehicle system Σ^i for $i \in \{1, 2\}$. In the diagram, we have defined $\gamma^i := a^i u^i + b^i$.

map $h(z^i) = [z_1^i - d_1, z_1^i + d_1] \times [z_2^i - d_2, z_2^i + d_2]$ (a continuous set-valued function), in which z^i is a pair of position/speed measurements assumed to be continuous in time, d_1 models uncertainty on the position measurement, and d_2 models uncertainty on the speed measurement. While d_2 is practically close to zero as the on-board speed measurements are quite accurate, d_1 can be quite large due to GPS positioning error. One can verify that systems Σ^i are order preserving systems, and the differential inclusion generated by all inputs is Marchaud.

The corresponding discrete-time dynamical system with time step ΔT is given by

$$\begin{aligned} x_1^i[n+1] &= x_1^i[n] + \Delta T x_2^i[n] \\ x_2^i[n+1] &= x_2^i[n] + \Delta T \gamma^i, \end{aligned}$$

in which $\gamma^i = a^i u^i + b^i$ in the central mode of Figure 4 and $\gamma^i = 0$ in the right and left modes of the same figure.

The bad set **B** is constructed with the rectangle set $B = \,]L^1, H^1[\, \times \,]L^2, H^2[$.

7.1 The Cooperative Case

In the cooperative case, we have that $u = (u^1, u^2) = \varphi_{\text{coop}}(\omega, \delta) = (\omega^1, \omega^2)$, that is, both of the agents are controlled and $(u^1, u^2) \in \Omega = [\omega_L^1, \omega_H^1] \times [\omega_L^2, \omega_H^2]$. We implement the algorithms of Section 6

to compute the restricted capture sets C_{ω_L} and C_{ω_H}. Figure 5 shows snapshots in the position plane of the trajectory of the set $[\wedge\hat{x}, \vee\hat{x}]$ for the closed-loop system. As soon as the set $[\wedge\hat{x}, \vee\hat{x}]$ hits the intersection of the two restricted capture sets C_{ω_L} and C_{ω_H}, the safety control acts and, as a result, set $[\wedge\hat{x}, \vee\hat{x}]$ slides along the boundary of the capture set until it passes **B**. Note that the sets C_{ω_L} and C_{ω_H} are each four dimensional. The plots of Figure 5 show slices of such sets in the position plane corresponding to the value of the current speeds.

7.2 The Competitive Case

In the competitive case, we have that $u = (u^1, u^2) = \varphi(\omega, \delta) = (\omega^1, \delta^2)$, that is, the first agent is controlled while the second one is not and $(u^1, u^2) \in [\omega_L^1, \omega_H^1] \times [\delta_L^2, \delta_H^2]$. We implement the algorithms of Section 6 to compute the restricted capture sets C_{ω_L} and C_{ω_H}. Figure 6 shows snapshots in the position plane of the trajectory of the set $[\wedge\hat{x}, \vee\hat{x}]$ for the closed-loop system.

7.3 The Combined Case: Experimental Results

In order to show the suitability of the proposed algorithms for real-time applications, we implemented the algorithms on the in-scale roundabout test-bed shown in Figure 7. The vehicles are equipped with an on-board computer running Linux Fedora core, wireless (802.11b), speed and position sensors, and a motion controller that translates desired torque commands for the wheels into a PWM signal applied to the DC motor. This guarantees that the vehicle responds to torque commands (calculated in the on-board computer) through a second order dynamics of the type of equation (15). For a detailed description of the vehicles, the reader is referred to [48]. The dynamical parameters for each vehicle were experimentally determined and resulted in the longitudinal dynamics model $\ddot{p}_i = a^i\tau^i + b^i + \mathcal{D}^i = f_2^i((p_i, \dot{p}_i), \varphi(\tau^i, \mathcal{D}^i))$, in which $\tau^i \in [0, 100]$ is the percentage torque control command applied to the wheels from the motor, $a^1 = 1.20$ cm/sec^2, $b^1 = -0.90$cm/sec^2, $a^2 = 1.26$ cm/sec^2, $b^2 = -1.15$ cm/sec^2, $\mathcal{D}^1 \in [0.6, 19.1]$ cm/sec^2, and $\mathcal{D}^2 \in [0.85, 24.85]$ cm/sec^2. A torque command of 100% corresponds to a torque of 0.09 N m. The terms $\mathcal{D}^i$ incorporate uncertainty that has been added to the model to take into account the parameter identification error. The limits on the speeds are taken as $v_{max} = 80$ cm/sec and $v_{min} = 25$ cm/sec. The speeds v_{max} and v_{min} given in the guard conditions in Figure 4 are maintained through the employment of a proportional derivative (PD) speed control. The longitudinal dynamics model corresponds to a game model $\mathscr{G}$ in which $u^i = \varphi^i(\omega^i, \delta^i) = (\frac{\omega^1+\delta^1}{2}, \frac{\omega^2+\delta^2}{2})$ with $\omega^i \in [0.0, 200.0]$, $\delta^1 \in [0.6, 19.1] \, 2/a^1$, and $\delta^2 \in [0.85, 24.85] \, 2/a^2$.

Vehicle control has two main components: maintaining the vehicles on the corresponding roundabout paths and applying the appropriate control torques ω to the longitudinal dynamics to prevent collisions at point C (Figure 7). In general, the longitudinal and lateral dynamics of a vehicle are coupled. However, since the radii of the paths are much greater than the length of the vehicles and the speeds are low, it is possible to assume low coupling. This allows us to decouple the path following task, using a steering control input, from the longitudinal dynamics control, using the torque control input ω.

When no special torque command is required to guarantee safety (the last case of the control map in Theorem 4), a cruise control algorithm comes into effect to maintain the vehicle speeds about pre-defined set points. For the roundabout implementation, vehicle 1 tracks a speed of 0.4 m/s, while vehicle 2 tracks a speed of 0.5 m/s. A PD controller is employed for this tracking task. These speeds were selected such that the vehicles would be able to accelerate and decelerate as much as possible
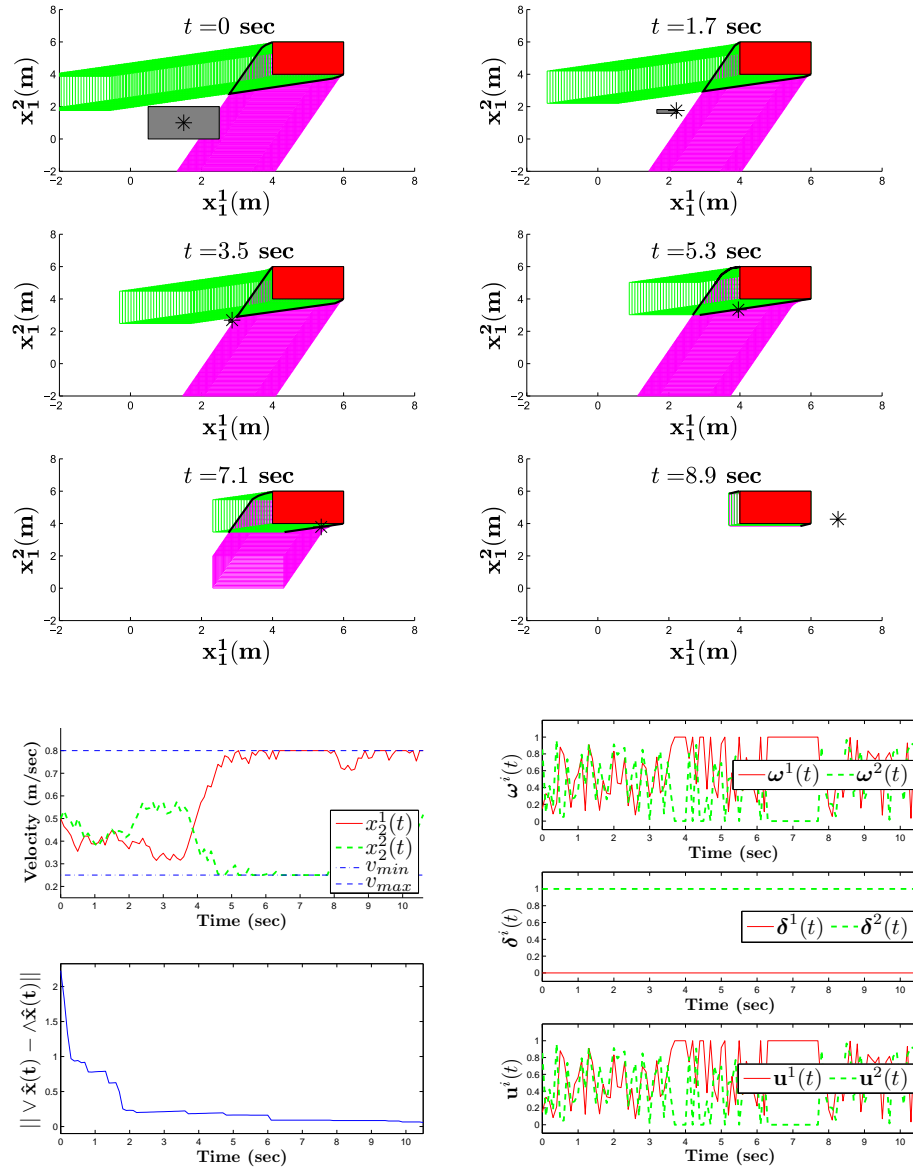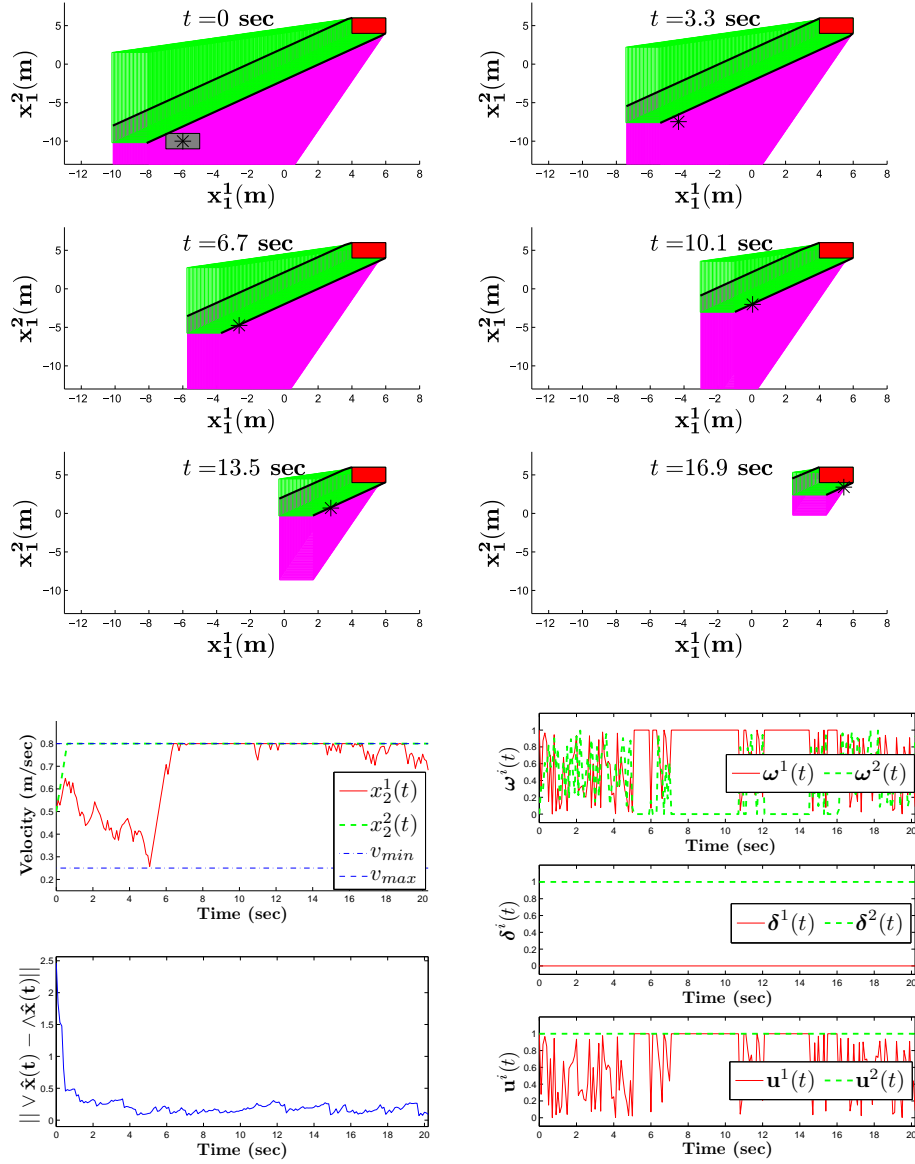
Figure 5: The cooperative case. The above plots depict snapshots of the dynamic evolution of the closed-loop system. The system considered has $a^i = 1$ and $b^i = -.5$ for $i \in \{1, 2\}$, with $v_{min} = .25$m/sec and $v_{max} = .8$m/sec. We choose $\Delta T = .1$ sec, $\mathbf{B} =]4, 6[\times\mathbb{R}\times]4, 6[\times\mathbb{R}$, $\Omega = [0, 1] \times [0, 1]$, $x_0 = (1.5, .5, 1, .5)$, $\hat{x}_0 = [.5, 2.5] \times [.4, .6] \times [0, 2] \times [.4, .6]$. The measurements z are generated randomly with a uniform probability distribution in the interval $[x(t) - (1, .1, 1, .1), x(t) + (1, .1, 1, .1)]$ so that $h(z) = [z - (1, .1, 1, .1), z + (1, .1, 1, .1)]$. The grey box represents the projection of $\hat{x}(t)$ onto the (x_1^1, x_1^2) plane, with the black asterisk representing the state of the system projected onto the (x_1^1, x_1^2) plane. The red box represents the projection of $\mathbf{B}$ onto the (x_1^1, x_1^2) plane, the slice of $\mathcal{C}_{\boldsymbol{\omega}_\mathcal{L}}$ corresponding to the current speeds is shown in green and the slice of $\mathcal{C}_{\boldsymbol{\omega}_\mathcal{H}}$ corresponding to the current speeds is shown in purple. Plots of the velocities, controls, disturbances, estimation error $\| \wedge \hat{x} - \vee \hat{x} \|$, and inputs are depicted in the lower panels.

while staying in the speed range enforced by the speed limiter. The range of speeds was selected based on the geometry of the roundabout such that the capture set $\mathcal{C}$ does not extend beyond the reference

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Figure 6: The competitive case. The above plots depict snapshots of the dynamic evolution of the closed-loop system. The system considered has $a^i = 1$ and $b^i = -.5$ for $i \in \{1, 2\}$, with $v_{min} = .25$m/sec and $v_{max} = .8$m/sec. We choose $\Delta T = .1$ sec, $\mathbf{B} =]4, 6[\times \mathbb{R} \times]4, 6[\times \mathbb{R}$, $\Omega = [0, 1] \times [0, 1]$, $\Delta = [0, 1] \times [0, 1]$, $x_0 = (-6, .5, -10, .5)$, $\hat{x}_0 = [-7, -5] \times [.4, .6] \times [-11, -12] \times [.4, .6]$. The measurements z are generated randomly with a uniform probability distribution in the interval $[x(t) - (1, .1, 1, .1), x(t) + (1, .1, 1, .1)]$ so that $h(z) = [z - (1, .1, 1, .1), z + (1, .1, 1, .1)]$. The grey box represents the projection of $\hat{x}(t)$ onto the (x_1^1, x_1^2) plane, with the black asterisk representing the state of the system projected onto the (x_1^1, x_1^2) plane. The red box represents the projection of $\mathbf{B}$ onto the (x_1^1, x_1^2) plane, the slice of $C_{\omega_{\mathcal{L}}}$ corresponding to the current speeds is shown in green and the slice of $C_{\omega_{\mathcal{H}}}$ corresponding to the current speeds is shown in purple. Plots of the velocities, controls, disturbances, estimation error $\| \wedge \hat{x} - \vee \hat{x} \|$, and inputs are depicted in the lower panels.

point on either path. If this were not the case, the vehicles may apply control to avoid the bad set on the first pass, only to end up in the capture set for the second pass, thus making it impossible to avoid
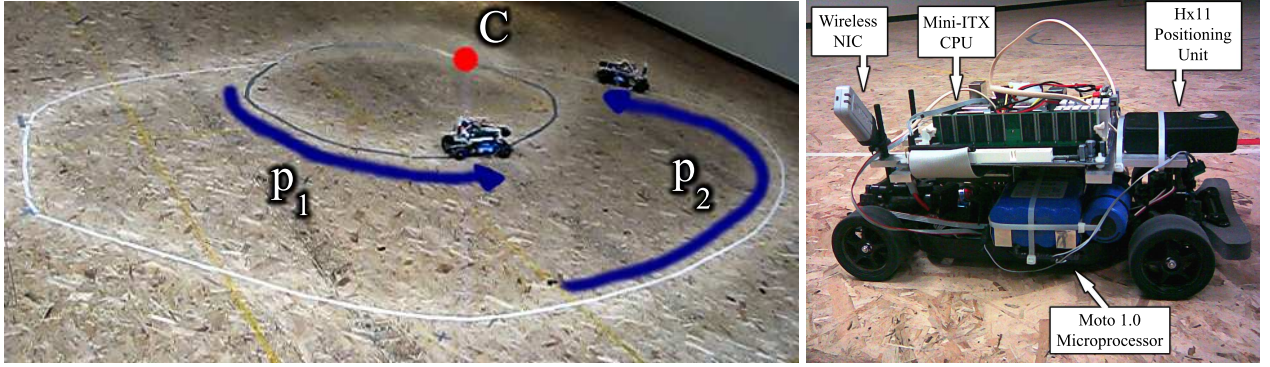
Figure 7: Roundabout test-bed (left). The longitudinal displacements of the vehicles with respect to a reference point along their corresponding paths are indicated by p_1 and p_2. The bad set **B** is a disk about point C. The vehicles (right).
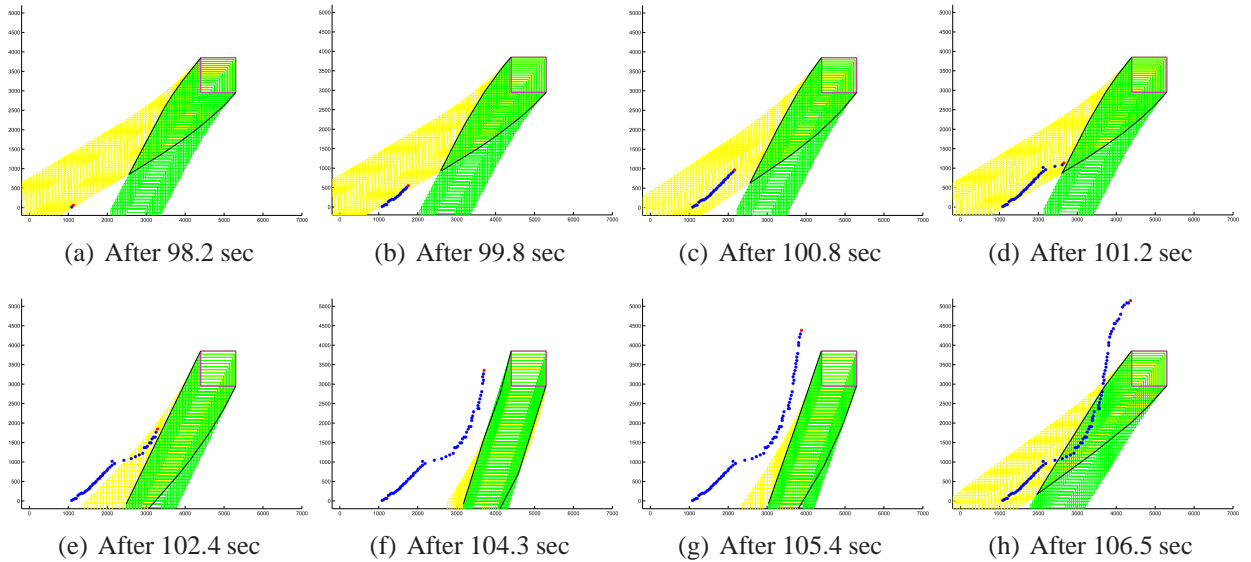
a collision.

(a) After 98.2 sec	(b) After 99.8 sec	(c) After 100.8 sec	(d) After 101.2 sec

(e) After 102.4 sec	(f) After 104.3 sec	(g) After 105.4 sec	(h) After 106.5 sec

Figure 8: Experiment data showing the trajectory in the position plane of the vehicles configuration as it approaches a potential collision scenario. The red box is the projection of **B** in the position plane. In each panel, the green set represents a slice of the four dimensional set $C_{\omega_{\mathcal{H}}}$ corresponding to the current vehicles speeds. The yellow set represents a slice of the four dimensional set $C_{\omega_{\mathcal{L}}}$ corresponding to the current vehicles speeds. The red dot indicates the current vehicles positions. Control is applied at (d) to avoid the capture set, and the vehicles resume normal operation after passing the bad set (in (g) and (h)). The capture set slices are updated at every iteration on the basis of the vehicles speeds.

Figure 8 illustrates the trajectory of the vehicle configuration projected onto the position plane, when avoiding a collision in one instance of the collision avoidance algorithm. The sets $C_{\omega_{\mathcal{L}}}$ and $C_{\omega_{\mathcal{H}}}$ are four dimensional. In the figure, we show the slices of these sets in the position plane corresponding to the current speeds of the vehicles.

8 Conclusions and Future Work

Since the dynamic feedback problem for general hybrid systems with imperfect state information is prohibitive, we focused on a restricted class of systems, which is still relevant for modeling a number of application scenarios. In particular, we focused on a class of hybrid systems with order preserving dynamics. For this class of systems, we have presented an explicit solution to the safety control problem with imperfect state information. We have provided linear complexity discrete-time algorithms for computing this solution. We have shown the application of these algorithms to a two-vehicle collision avoidance scenario at a traffic intersection. The experimental results confirm the suitability of these algorithms for fast real-time computation.

There are a number of future research avenues to be explored in the context of imperfect state information. Specifically, we will consider the extension of this approach to hybrid dynamics with discrete state memory. Also, this work has focused on two-player games. We seek to extend it to multi-agent games and apply it to multi-vehicle collision avoidance scenarios at traffic intersections. In this case, we expect that the two-vehicle collision avoidance algorithm will be employed as a primitive to construct the solution of the multi-vehicle collision avoidance problem.

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9 Appendix: Proof of Lemmas and Propositions

9.1 Proof of Lemma 1.

Before giving the proof, we need the following intermediate results.

Proposition 2. *Consider order preserving game structure $\mathscr{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$ and let $x \in X$, $\omega \in S(U)$, $\delta \in S(\Delta)$ and $\gamma \in C^0(I, \mathbb{R}^2)$ o.p.c. where $x_1^1 \leq \max \tau_1(\gamma(I))$. Then, we have that either $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \succ \gamma(I)$ or $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \prec \gamma(I)$ if and only if $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) = \emptyset$.*

Proof. ($\Rightarrow$) Follows from the definition of the $\prec$ relation.

($\Leftarrow$) Suppose $\{\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \succ \gamma(I)$ or $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \prec \gamma(I)\}$ does not hold. The hypothesis $\phi_1^1(0, x^1, \varphi^1(\omega^1, \delta^1)) \leq \sup \tau_1(\gamma(I))$ and condition (iii) of Definition 4 imply that there exist $\alpha^1, \alpha^2 \in I$ and $t_1, t_2 \in \mathbb{R}_+$ such that $\phi_1(t_1, x, \varphi(\omega, \delta)) \leq \gamma(\alpha^1)$ and $\phi_1(t_2, x, \varphi(\omega, \delta)) \geq \gamma(\alpha^2)$. For simplifying notation, let $\zeta(t) := \phi_1(t, x, \varphi(\omega, \delta))$. Without loss of generality, assume $\alpha^1 \leq \alpha^2$, define $\chi \in \mathbb{R}^2$ where $\chi_1 := \min\left\{\gamma_1(\alpha^1), \gamma_1(\alpha^2)\right\}$ and $\chi_2 := \min\{\phi_1^2(t_1, x^2, \varphi^2(\omega^2, \delta^2)), \gamma_2(\alpha^2)\}$. Next, we define $\Gamma_{12} :=$
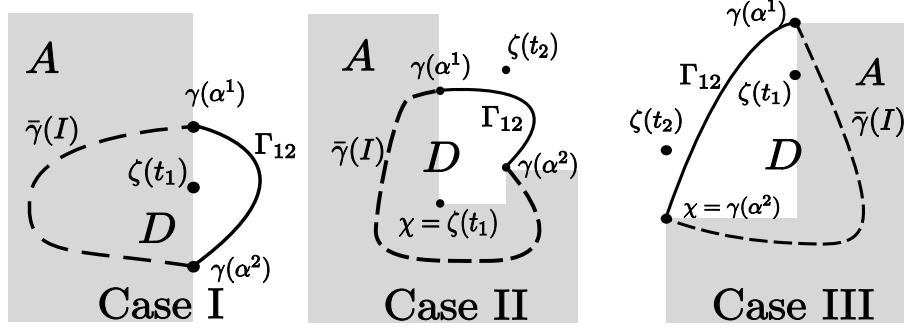
Figure 9: Three Cases.

$\gamma([\alpha^1, \alpha^2])$. By the construction of χ, we have that $\gamma(\alpha^1), \gamma(\alpha^2) \in \text{Cone}_+(\chi)$, which implies that $\Gamma_{12} \subset \text{Cone}_+(\chi)$ by the definition of o.p.c. We now consider the three possible cases: (Case I) $t_1 = t_2$, (Case II) $t_1 < t_2$, and (Case III) $t_1 > t_2$.

(Case I) Suppose $t_1 = t_2$, implying $\gamma(\alpha^2) \le \zeta(t_1) \le \gamma(\alpha^1)$. Consider the open half space $A := \sim \text{Cone}_{\{e_1\}}(\chi) \subset \mathbb{R}^2$ which is trivially path connected, and the set $\tilde{A} := A \cup \gamma(\alpha^1) \cup \gamma(\alpha^2)$. The set $\tilde{A}$ is also path connected, implying the existence of a path $\bar{\gamma} \in C^0(I, \tilde{A})$ such that $\bar{\gamma}(0) = \gamma(\alpha^1)$ and $\bar{\gamma}(1) = \gamma(\alpha^2)$ where $\bar{\gamma}$ is simple. Since $\Gamma_{12} \subset \text{Cone}_+(\chi)$, and $\text{Cone}_{\{-\hat{e}_1\}}(\chi) \cap \text{Cone}_+(\chi) = \emptyset$ by definition of the cone, we must have that $A \cap \Gamma_{12} = \emptyset$. This implies that $\bar{\gamma}(I)$ only intersects Γ_{12} at $\bar{\gamma}(0)$ and $\bar{\gamma}(1)$, allowing us to re-parameterize $\bar{\gamma}(I) \cup \Gamma_{12}$ with a simple closed curve (see Figure 9).

This simple closed curve, by the Jordan Curve Theorem [38], partitions $\mathbb{R}^2$ into two sets, D bounded and $\sim D$ unbounded. By construction, D is such that $\zeta(t_1) \in D$ and $\partial D = \Gamma_{12} \cup \bar{\gamma}(I)$. Condition (iii) of Definition 4 implies that $\|\zeta(t)\| \to \infty$ as $t \to \infty$. Thus, $\zeta([\bar{t}, \infty)) \cap \partial D$ must be non-empty because D is a bounded set. Since condition (iii) of Definition 4 implies that $\zeta([\bar{t}, \infty)) \cap \tilde{A}$ is empty and $\bar{\gamma}(I) \subset \tilde{A}$, we must have that $\zeta([\bar{t}, \infty)) \cap \Gamma_{12} \ne \emptyset$. This in turn implies $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) \ne \emptyset$.

(Case II) Suppose $t_1 < t_2$. This along with condition (ii) of Definition 4 implies that $\gamma_1(\alpha^1) < \gamma_1(\alpha^2)$. We assume that $\zeta(t_1) \le \gamma(I)$ and $\zeta(t_2) \ge \gamma(I)$, otherwise we would be back to (Case I). Define the sets $S_1 := \text{Cone}_{\{\hat{e}_1, -\hat{e}_2\}}(\gamma(\alpha^2))$ and $S_2 := \text{Cone}_+(\chi)$. Define $A := \mathring{S}_1 \cup (\sim S_2)$ and $\tilde{A} := A \cup \gamma(\alpha^1) \cup \gamma(\alpha^2)$. Since γ is an o.p.c. path, $\Gamma_{12} \subset \text{Cone}_+(\chi)$ and $\Gamma_{12} \cap S_1 = \emptyset$, we must have that $\Gamma_{12} \cap A = \emptyset$. The set $\tilde{A}$ is path connected, implying the existence of $\bar{\gamma} \in C^0(I, \tilde{A})$ with $\bar{\gamma}(0) = \gamma(\alpha^1)$, $\bar{\gamma}(1) = \gamma(\alpha^2)$ and $\bar{\gamma}$ simple. Since $A \cap \Gamma_{12} = \emptyset$, $\bar{\gamma}(I) \cup \Gamma_{12}$ can be re-parameterized with a simple closed curve (see Figure 9). This curve, by the Jordan Curve Theorem, forms a bounded set D, where $\zeta(t_1) \in D$ by construction. Condition (ii) and (iii) of Definition 4 along with the decoupling of the dynamics imply that $\zeta([t_1, \infty]) \cap A = \emptyset$ and $\zeta([t_1, \infty]) \cap \partial D \ne \emptyset$. Since $\bar{\gamma} \subset A$, we have that $\zeta([t_1, \infty]) \cap \Gamma_{12} \ne \emptyset$. Therefore, $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) \ne \emptyset$.

(Case III) Suppose $t_2 < t_1$, which along with condition (iii) of Definition 4 implies that $\gamma_1(\alpha^2) < \gamma_1(\alpha^1)$. We assume that $\zeta(t_1) \le \gamma(I)$ and $\zeta(t_2) \ge \gamma(I)$, otherwise we would be back to (Case I). Define the sets $P := \text{Cone}_{\{e_1\}}(\chi)$, $R := \text{Cone}_{\{e_1, -e_2\}}(\gamma(\alpha^1))$, $H := \text{Cone}_{(+)}(\chi) \backslash \mathring{R}$, $A := P \backslash H$, and $\tilde{A} := A \cup \gamma(\alpha^1) \cup \gamma(\alpha^2)$. The set $\tilde{A}$ is path connected, implying the existence of $\bar{\gamma}$ where $\bar{\gamma} \in C^0(I, \tilde{A})$ with $\bar{\gamma}(0) = \gamma(\alpha^1)$, $\bar{\gamma}(1) = \gamma(\alpha^2)$ and $\bar{\gamma}$ simple. Observe that $A \cap \Gamma_{12} = \emptyset$, thus $\bar{\gamma}(I) \cup \Gamma_{12}$ can be re-parametrized with a simple closed curve. We invoke the Jordan Curve Theorem to construct the bounded set D, where $\zeta(t_1) \in D$ by construction (Figure 9). By construction, we also have that $\zeta(t_2) \notin D$. Thus, the uniform continuity of the flow with respect to time implies $\zeta([t_2, t_1]) \cap \partial D \ne \emptyset$. Condition (iii) of Definition 4 implies that $\zeta([t_2, t_1]) \subset H$, thus implying $\zeta([t_2, t_1]) \cap \tilde{A} = \emptyset$. Since $\partial D = \Gamma_{12} \cup \bar{\gamma}(I)$ and

$\bar{\gamma}(I) \subset \tilde{A}$, we must have $\zeta([t_2, t_1]) \cap \Gamma_{12} \neq \emptyset$. This implies that $\zeta([t_2, t_1]) \cap \gamma(I) \neq \emptyset$, giving the desired result $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) \neq \emptyset$.

Therefore, we have shown for each case $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) \neq \emptyset$, completing the proof. $\qquad\square$

Proposition 2 states that the flow ϕ generated from the initial condition x, controlled input ω and disturbance δ can avoid an o.p.c. path γ in the (x_1^1, x_1^2) subspace if and only if the trajectory of ϕ_1 lies above $\gamma(I)$ or if the trajectory of ϕ_1 lies below $\gamma(I)$. Another intermediate result is needed before stating the proof of Lemma 1.

Proposition 3. *Consider order preserving game structure $\mathcal{G} = (\Sigma, \Omega, \Delta, \varphi, \mathbf{B})$, $x \in X$, $\omega \in S(U)$ and $\gamma \in C^0(I, \mathbb{R}^2)$ o.p.c. with $x_1^1 \leq \max \tau_1(\gamma(I))$. If $\bigcup_{\delta \in S(\Delta)} \phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) = \emptyset$, then either $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{L}})) > \gamma(I)$ or $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{H}})) < \gamma(I)$.*

Proof. The assumption that $\gamma(I) \cap \bigcup_{\delta \in S(\Delta)} \phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) = \emptyset$ implies
(a) $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{L}})) \cap \gamma(I) = \emptyset$ and (b) $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{H}})) \cap \gamma(I) = \emptyset$. From Proposition 2, we have that (a) implies either

$$\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{L}})) > \gamma(I) \qquad (16)$$

$$\text{or} \qquad \phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{L}})) < \gamma(I). \qquad (17)$$

Similarly, Proposition 2 along with (b) implies either

$$\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{H}})) > \gamma(I) \qquad (18)$$

$$\text{or} \qquad \phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_{\mathcal{H}})) < \gamma(I). \qquad (19)$$

If (16) is satisfied, we immediately obtain the result. Similarly, if (17) and (19) are satisfied the result also follows. Therefore, we are left with showing that relations (17) and (18) are not both possible. By contradiction, assume they are both possible and define the constant signals $\delta_L^2(t) := \delta_L^2$, $\delta_H^2(t) := \delta_H^2$, $\delta_L^1(t) := \delta_L^1$, and $\delta_H^1(t) := \delta_H^1$ for all $t \in \mathbb{R}_+$. Then, there is $(\alpha^1, \alpha^2) \in \gamma(I)$, $t_a > 0$, and $t_b > 0$ such that $\phi_1^2(t_a, x^2, \varphi^2(\omega^2, \delta_L^2)) < \alpha^2$ and $\phi_1^2(t_b, x^2, \varphi^2(\omega^2, \delta_H^2)) > \alpha^2$. Since $\phi_1^1(t_a, x^1, \varphi^1(\omega^1, \delta_H^1)) = \alpha^1 = \phi_1^1(t_b, x^1, \varphi^1(\omega^1, \delta_L^1))$, the order preserving property of φ in its arguments imply that $t_a \leq t_b$. For fixed x^2 and ω^2, define the function $\Phi_1^2 : [t_a, t_b] \times S(\Delta^2) \to \mathbb{R}$ by $\Phi_1^2(t, \delta^2) := \phi_1^2(t, x^2, \varphi^2(\omega^2, \delta^2))$. This is a continuous function from a connected metric space into the reals. Therefore, we can apply the intermediate value theorem to state that there is a pair $\bar{t} \in [t_a, t_b]$ and $\bar{\delta}^2 \in S(\Delta^2)$ such that $\Phi_1^2(\bar{t}, \bar{\delta}^2) = \alpha^2$.

Property (iii) of Definition 4 further implies that the ordering $\phi_1^1(\bar{t}, x^1, \varphi^1(\omega^1, \delta_H^1)) > \alpha^1$ and ordering $\phi_1^1(\bar{t}, x^1, \varphi^1(\omega^1, \delta_L^1)) < \alpha^1$ must hold. For fixed x^1 and ω^1, define the map $\Phi_1^1 : S(\Delta^1) \to \mathbb{R}$ by $\Phi_1^1(\delta^1) := \phi_1^1(\bar{t}, x^1, \varphi^1(\omega^1, \delta^1))$. This is a continuous function from a connected metric space to the reals, therefore we can apply again the intermediate value theorem to conclude that there is $\bar{\delta}^1 \in S(\Delta^1)$ such that $\Phi_1^1(\bar{\delta}^1) = \alpha^1$.

As a consequence, we have that $\phi_1(\bar{t}, x, \varphi(\omega, (\bar{\delta}^1, \bar{\delta}^2))) = (\alpha^1, \alpha^2) \in \gamma(I)$ for $(\delta^1, \delta^2) \in S(\Delta)$. This in turn contradicts the assumption that $\bigcup_{\delta \in S(\Delta)} \phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta)) \cap \gamma(I) = \emptyset$.

$\qquad\square$

Proposition 3 states that the flow ϕ generated from the initial condition x and controlled input ω will avoid an o.p.c. path γ in the (x_1^1, x_1^2) subspace if and only if the trajectory of ϕ_1 generated with the disturbance signal $\delta_{\mathcal{L}}$ lies above $\gamma(I)$ or if the trajectory of ϕ_1 generated with the disturbance signal $\delta_{\mathcal{H}}$ lies below $\gamma(I)$.
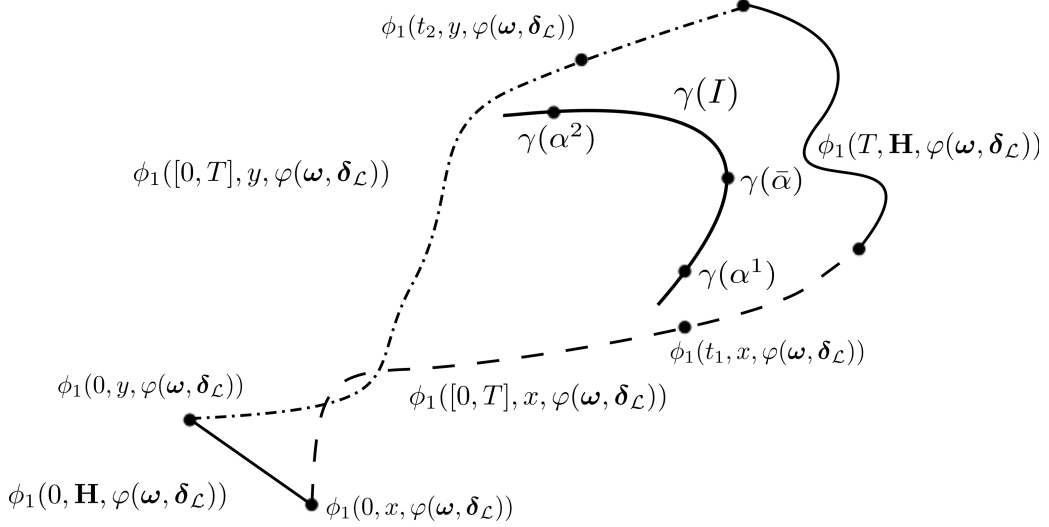
Figure 10: Geometry of $\phi_1(t, x, \varphi(\omega, \delta_\mathcal{H}))$ and $\phi_1(t, y, \varphi(\omega, \delta_\mathcal{L}))$.

Proof. (*Lemma 1*) ($\Leftarrow$) For every disturbance $\delta \in S(\Delta)$, we have that $\delta_\mathcal{L} \trianglelefteq \delta \trianglelefteq \delta_\mathcal{H}$. From Proposition 1, it follows that for every $x \in A$ and $t \in \mathbb{R}_+$, we have that $\phi(t, x, \varphi(\omega, \delta_\mathcal{L})) \precsim \phi(t, x, \varphi(\omega, \delta)) \precsim \phi(t, x, \varphi(\omega, \delta_\mathcal{H}))$. Therefore, the result follows directly from the assumption.

($\Rightarrow$) Suppose $\{\phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta_\mathcal{L})) > \gamma(I)$ or $\phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta_\mathcal{H})) < \gamma(I)\}$ does not hold. Then there must exist $x, y \in A$, $\alpha^1, \alpha^2 \in I$, and $t_1, t_2 > 0$ such that $\phi_1(t_1, x, \varphi(\omega, \delta_\mathcal{L})) < \gamma(\alpha^1)$ and $\phi_1(t_2, y, \varphi(\omega, \delta_\mathcal{H})) > \gamma(\alpha^2)$ (the relation is strict, otherwise the result is immediate). We assume that $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_\mathcal{L})) < \gamma(I)$, otherwise Proposition 2 implies that $\phi_1(\mathbb{R}_+, x, \varphi(\omega, \delta_\mathcal{L})) \cap \gamma(I) \neq \emptyset$. Likewise, Proposition 2 implies we must have $\phi_1(\mathbb{R}_+, y, \varphi(\omega, \delta_\mathcal{H})) > \gamma(I)$. Furthermore, unless $\phi_1(\mathbb{R}_+, y, \varphi(\omega, \delta_\mathcal{L})) > \gamma(I)$ is satisfied, the previous statement along with Proposition 3 implies that $\phi_1(\mathbb{R}_+, y, \varphi(\omega, \delta_\mathcal{L})) \cap \gamma(I) \neq \emptyset$. Figure 10 shows the resulting geometry of the flow. Let $\bar{\alpha} \in I$ be such that $\tau_1(\gamma(I)) \leq \tau_1(\gamma(\bar{\alpha}))$. Condition (iii) of Definition 4 leads to $x_1^1 < \phi_1^1(t_1, x^1, \varphi^1(\omega, \delta_\mathcal{L})) \leq \gamma_1(\bar{\alpha})$ and $y_1^1 < \phi_1^1(t_2, y^1, \varphi^1(\omega, \delta_\mathcal{L})) \leq \gamma_1(\bar{\alpha})$. Consider $H := \overline{\text{co}}\{x, y\} \subset A$, since convexity is preserved under projection [16], condition (iii) of Definition 4 implies there is $T > 0$ such that

$$\phi_1^1(0, \tau_1(H), \varphi^1(\omega^1, \delta_\mathcal{L}^1)) < \gamma_1(\bar{\alpha}) < \phi_1^1(T, \tau_1(H), \varphi^1(\omega^1, \delta_\mathcal{L}^1)). \tag{20}$$

We seek to show that $\gamma(\bar{\alpha}) \in \phi_1([0, T], H, \varphi(\omega, \delta_\mathcal{L}))$. Define $K := [0, T] \times H \subset \mathbb{R}_+ \times \mathbb{R}^{2n}$ and let $\Theta : K \to \mathbb{R}^2$ be the map defined by $\Theta(t, z) := \phi_1(t, z, \varphi(\omega, \delta_\mathcal{L}))$ for $(t, z) \in K$. We proceed by breaking this proof into three steps:

(i)   Construct from Θ a map $\psi : \mathbb{S}^1 \to \mathbb{S}^1$;

(ii)   Show that the degree of ψ is nonzero;

(iii)   Show that the degree of ψ being nonzero implies that $\gamma(\bar{\alpha}) \in \Theta(K)$.

(i) Denote the four corners of ∂K : $h_1 = (0, x)$, $h_2 = (T, x)$, $h_3 = (T, y)$, $h_4 = (0, y)$. Define the sets $A_1 := \overline{\text{co}}(\{h_1, h_2\}) \cup \text{co}(\{h_2, h_3\})$ and $A_2 := \overline{\text{co}}(\{h_3, h_4\}) \cup \overline{\text{co}}(\{h_4, h_1\})$. Consider the standard covering map of $\mathbb{S}^1$ $p : \mathbb{R} \to \mathbb{S}^1$, in which $p(z) := (cos(2\pi z), sin(2\pi z))$. Define the homeomorphism $f :$

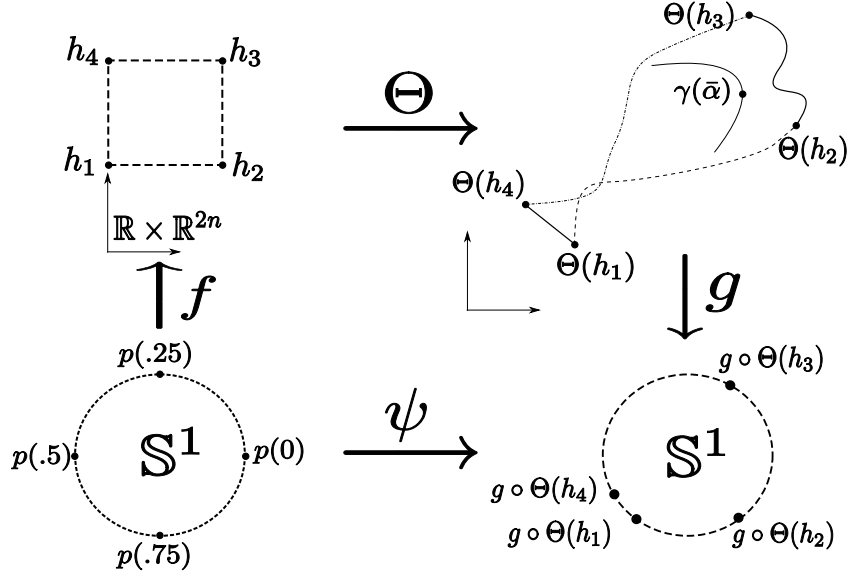Figure 11: The Mapping ψ.

(a) Commuting Diagram

(b) Four Quadrants of $\mathbb{S}^1$
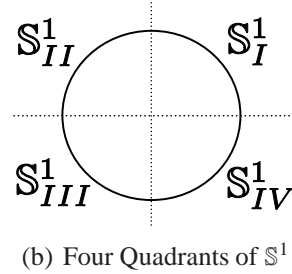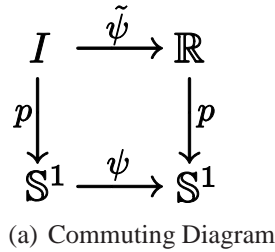
Figure 12: Tools used to find deg ψ.

$\mathbb{D}^1 \to K$, such that $f(p(0)) = h_1$, $f(p(.25)) = h_2$, $f(p(.5)) = h_3$, and $f(p(.75)) = h_4$. Since Θ is a continuous function, we have that $\Theta(\partial K)$ defines a closed curve. Assume that $\gamma(\bar{\alpha}) \notin \Theta(\partial K)$ and let $g : \mathbb{R}^2 \backslash \gamma(\bar{\alpha}) \to \mathbb{S}^1$ be the continuous map defined by

$$g(z) := \frac{z - \gamma(\bar{\alpha})}{\|z - \gamma(\bar{\alpha})\|}, \quad \forall z \in \mathbb{R}^2 \backslash \gamma(\bar{\alpha}). \tag{21}$$

Define $\psi \in C^0(\mathbb{S}^1, \mathbb{S}^1)$ as $\psi(x) := g \circ \Phi \circ f(x)$ for all $x \in \mathbb{S}^1$ (see Figure 11).

(ii) To compute the degree of ψ, we consider the lift $\tilde{\psi} : I \to \mathbb{R}$ where $p \circ \tilde{\psi} = \psi \circ p$ (see Figure 12(a)). The degree of ψ is defined as $\deg\psi := \tilde{\psi}(1) - \tilde{\psi}(0)$ (see [34] for details). We introduce the sets $\mathbb{S}^1_I := p([0, .25]), \mathbb{S}^1_{II} := p([.25, .5]), \mathbb{S}^1_{III} := p([.5, .75]), \mathbb{S}^1_{IV} := p([.75, 1])$ (see Figure 12(b)). Let $\kappa_1 := \tilde{\psi}(0)$ and note that $p(\kappa_1) = \psi(p(0)) = g(\Theta(h_1))$, which must be in $\mathbb{S}^1_{III}$, since $\Theta(h_1) < \gamma(\bar{\alpha})$. Let $\kappa_2 = \tilde{\psi}(.5)$ and note that $p(\kappa_2) = \psi(p(.5)) = g(\Theta(h_3))$. From (20) and condition (iii) of Definition 4, we have that $\gamma(\bar{\alpha}) < \Theta(h_3)$. This inequality along with the definition of g imply that $g(\Theta(h_3)) \in \mathbb{S}^1_I$. As a consequence, we have $p(\kappa_2) \in \mathbb{S}^1_I$, implying that $\kappa_1 \neq \kappa_2$.

We next show that $\kappa_2 > \kappa_1$. Since $\Theta \circ f(p([0, .5])) = \Theta(A_1)$, equation (20) along with condition (iii) of Definition 4 implies that $\Theta(A_1) \prec \gamma(\bar{\alpha})$. This implies that $\psi(p([0, .5])) = g(\Theta(A_1)) \subset \mathbb{S}^1_I \cup \mathbb{S}^1_{IV} \cup \mathbb{S}^1_{III}$. Therefore, if $\psi(p(\zeta))$ cannot enter $\mathbb{S}^1_{II}$ for all $\zeta \in [0, .5]$, then $\kappa_1 < \kappa_2$ by the definition of p.

Finally, let $\kappa_3 := \tilde{\psi}(1)$. We show that $\kappa_2 < \kappa_3$. Since $\Theta \circ f(p([.5, 1])) = \Theta(A_2)$, from (20) and Condition (ii) of Definition 4 we have that $\Theta(A_2) \succ \gamma(\bar{\alpha})$. This, along with Condition (iii) of Definition

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(4) implies that $\psi(p([.5,1])) = g(\Theta(A_2)) \subset \mathbb{S}^1_I \cup \mathbb{S}^1_{II} \cup \mathbb{S}^1_{III}$. Therefore, if $\psi(p(\zeta))$ cannot enter S^1_{IV} for all $\zeta \in [.5,1]$, then $\kappa_2 < \kappa_3$ from the definition of p.

We have shown that $\kappa_1 < \kappa_2 < \kappa_3$. As a consequence, $\deg \psi = \tilde{\psi}(1) - \tilde{\psi}(0) = \kappa_3 - \kappa_1 \neq 0$.

(iii) Now suppose we extend the map ψ to $\bar{\psi} \in C^0(\mathbb{D}^1, \mathbb{S}^1)$, where $\bar{\psi}(x) := g \circ \Theta \circ f(x)$ for all $x \in \mathbb{D}^1$. By Lemma 3.5.7 in [34], if a continuous function $h : \mathbb{S}^1 \to \mathbb{S}^1$ extends to a continuous function $H : \mathbb{D}^1 \to \mathbb{S}^1$, then $\deg h$ must be zero. However, we found the degree of ψ to be non-zero, implying that ψ cannot extend to $\bar{\psi}$. Since $\Theta(f(\mathbb{D}^1))$ is well defined, we must have that $g(\Theta(f(\mathbb{D}^1)))$ is undefined. Since $g(z)$ is defined for all $z \in \mathbb{R}^2 \backslash \gamma(\bar{\alpha})$, we must have that $\gamma(\bar{\alpha}) \in \Theta(f(\mathbb{D}))$. This implies that $\gamma(\bar{\alpha}) \in \Theta(K) = \phi_1([0,T], H, \varphi(\omega, \delta_{\mathcal{L}})) \subset \bigcup_{\delta \in S(\Delta)} \phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta))$. Therefore, $\bigcup_{\delta \in S(\Delta)} \phi_1(\mathbb{R}_+, A, \varphi(\omega, \delta)) \cap \gamma(I) \neq \emptyset$.

$\square$