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Influence of feedback on the stochastic evolution of simple climate systems

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We consider the dynamical evolution of a simple climate system that describes the average temperature of the Earth's atmosphere owing to radiative forcing and coupling to a positive feedback variable such as the concentration of greenhouse gases in the presence of fluctuations. Analysing the resulting stochastic dynamical system shows that, if the temperature relaxes rapidly relative to the concentration, the time-dependent and stationary probability density functions (pdfs) for the temperature rise possess a fat tail. In contrast, if the feedback variable relaxes rapidly relative to the temperature, the pdf has no fat tail, and, instead, the system shows critical slowing down as the singular limit of positive feedback is approached. However, if there is uncertainty in the feedback variable itself, a fat tail can reappear. Our analysis may be generalized to more complex models with similar qualitative results. Our results have policy implications: although fat tails imply that the expectation of plausible damage functions is infinite, the pdfs permit an examination of the trade-off between reducing emissions and reducing the positive feedback gain.

Keywords: stochastic climate system; feedback; fat tails

1. Introduction and model

Global warming couples a variety of effects on multiple space and time scales, from fast atmospheric circulation on a daily or weekly time scale to slow, large-scale ocean circulation that varies on time scales of centuries to millennia. It is also greatly influenced by feedback mechanisms that include variations in polar ice-cap extent, desertification, water vapour concentration and cloud feedback, to name just a few causes (Graves *et al.* 1993; Scheffer *et al.* 2006; Torn & Hart 2006). The combination of these multi-scale effects and feedback leads to large uncertainties in the outcome of climate models, so that the natural language for climate variables, such as the global mean temperature, is its probability density function (pdf), rather than its deterministic value.

Understanding the effects of uncertainty from the properties of the pdf and its temporal evolution is thus of crucial importance. One approach to obtaining the pdf is via global climate models (GCMs) that are capable of

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simulating some of the highly complex radiative, chemical, thermodynamic and hydrodynamic processes. However, sheer practicality means that GCMs leave out effects associated with fluctuations that occur on fast time scales. Here, we analyse the effect of a single feedback mechanism on the dynamical evolution of the temperature, in a simple stochastic climate model, in order to sharpen the understanding of the role that these fluctuating effects play.

The simple model we adopt consists of a pair of coupled stochastic dynamical equations for the time evolution of the temperature $T(t)$ (Tung 2007) and the concentration of greenhouse gases (GHGs), $c(t)$:

$$\rho C_p H \frac{dT(t)}{dt} = R(c) - \sigma(1 - g(c))T(t)^4 + \hat{f}_T(t), \quad (1.1)$$

and

$$\tau_c \frac{dc(t)}{dt} = -c(t) + \hat{q}(t) + \hat{f}_c(t). \quad (1.2)$$

Here, equation (1.1) is the radiative energy balance equation averaged over the entire Earth with an atmosphere that has density ρ , specific heat C_p and depth H . σ is the Stefan–Boltzmann constant and $g(c)$ is a feedback parameter that modifies the Stefan–Boltzmann law in a concentration-dependent manner. The atmosphere heats or cools because of an imbalance between (i) the net solar influx $R(c)$, which accounts for the incoming solar radiation and the effects of albedo and atmosphere in a composition-dependent manner, characterized in terms of the average concentration $c(t)$ of GHGs, and (ii) the radiation from Earth, accounted for by a modification of the Stefan–Boltzmann law that introduces feedback in the presence of GHGs. Equation (1.2) is the scaled equation for the evolution of the concentration of GHGs, with $\hat{q}(t)$ being the addition rate of GHG per unit volume to the atmosphere. The terms $\hat{f}_T(t)$ and $\hat{f}_c(t)$ are independent random forcing terms, assumed to be Gaussian, that represent fast, uncorrelated, changes in temperature and GHG emission rate, respectively. These coupled stochastic dynamical equations are ubiquitous in modelling physical phenomena, and it may be pertinent to point to a formal resemblance of the current system to intermittency in single-molecule spectroscopy (Wang & Wolynes 1995, 1999; Cao & Silbey 2008).

To understand the effects of climate change driven by changes in concentration of GHGs, we consider small deviations from the equilibrium temperature T_0 consistent with a pre-industrial equilibrium concentration of GHGs, c_0 . Expanding about this state, i.e. substituting $\Delta T(t) = T(t) - T_0$ and $\Delta c(t) = c(t) - c_0$, and keeping only leading order terms leads to a scaled form of the coupled climate model, equations (1.1) and (1.2) (see appendix A),

$$\frac{d\Delta T(t)}{dt} = \frac{\lambda_0 \Delta R_p(c(t))}{\tau_T} - \frac{[1 - \gamma \Delta c]}{\tau_T} \Delta T(t) + f_T(t) \quad (1.3)$$

and

$$\frac{d\Delta c(t)}{dt} = -\frac{1}{\tau_c} \Delta c(t) + q(t) + f_c(t), \quad (1.4)$$

where $g = g_0 + g'_0 \Delta c$, $\gamma = g'_0 / (1 - g_0)$ with $g_0 = g(c_0)$, $g'_0 = dg/dc|_{c=c_0}$ and the incremental radiative forcing, $\lambda_0 \Delta R_p$, to the lowest order in the concentration deviation, is $\lambda_0 \Delta R_p = \lambda_0 R'_p(c) \Delta c$, with λ_0 a constant. Equation (1.3) couples

the change in the equilibrium temperature via feedback to the change in GHG concentration only, although, in general, feedback may also depend on water vapour concentration, clouds, soil conditions, etc., each of which may evolve separately. Here, we focus on the influence of GHG concentration on the feedback and so assume $g_0 < 1$. Then, the feedback parameter γ couples the GHG concentration to the temperature, with $\gamma > 0$ ($\gamma < 0$) corresponding to positive (negative) feedback.

2. Analysis of model

There are two time scales in the problem defined by equations (1.3) and (1.4), set by the relaxation times, τ_T and τ_c , for the equilibration of temperature and concentration. Both relaxation times are determined by loss factors, such as transfer of heat or mass to the ocean, and by dissipation arising from the fluctuations. The potential for vastly different time scales for the relaxation of temperature and GHG concentration, both of which can themselves vary over many orders of magnitude, suggests that the dynamics of the simple climate system, equations (1.3) and (1.4), is best treated via an analysis of various limiting cases.

We first consider the case of fast temperature relaxation, i.e. $\tau_T/\tau_c \ll 1$. On the time scale $\tau_c \sim t \gg \tau_T$, radiative equilibrium is established. Incremental radiative forcing, $\Delta R_p(c)$, causes a temperature change $\Delta T = \lambda_0 \Delta R_p(c)/(1 - \gamma \Delta c)$. For simplicity, we consider the case of linear forcing, $\Delta R_p(c) = \alpha \Delta c$, so that

$$\frac{\Delta T}{\Delta T_0} = \frac{|\gamma| \Delta c}{(1 - \gamma \Delta c)}, \quad \text{with } \Delta T_0 = \lambda_0 \alpha |\gamma|^{-1}. \quad (2.1)$$

It is useful to rewrite the above relations in terms of the equilibrium climate sensitivity parameter using the logarithmic derivative or elasticity, $S_0 = d \ln T_0(c)/d \ln c$, following Torn & Hart (2006), and the sensitivity of the fluctuations, $S_\Delta = d \ln \Delta T(\Delta c)/d \ln \Delta c$.¹ When the radiative forcing is linear, $S_0 = (1 + \gamma)/4$, indicating greater sensitivity for larger positive feedback values, but the sensitivity to fluctuations is still greater, since $S_\Delta = \gamma \Delta c (1 - \gamma \Delta c)^{-1}$. In the absence of feedback, $S_0 = 1/4$, consistent with the Stefan–Boltzmann law, whereas $S_\Delta = 0$ since there is no incremental change in the temperature in the absence of GHG fluctuations.

For positive feedback, $\gamma > 0$, we may write the above relation in two equivalent forms: $\Delta T/\Delta T_0 = \gamma \Delta c/(1 - \gamma \Delta c)$ or $\gamma \Delta c = \Delta T/(\Delta T_0 + \Delta T)$. It is evident that these relations are valid only for $\Delta c < \gamma^{-1} = \Delta c_{\max}$; as this point is approached, the average atmospheric temperature diverges since heat loss vanishes so that $0 \leq \Delta T \leq \infty$ as $0 \leq \Delta c \leq \Delta c_{\max}$.

The concentration, $c(t)$, evolves on the slow time scale, i.e. $t \sim \tau_c \gg \tau_T$, according to equation (1.4). If the fluctuation $f_c(t)$ is Gaussian, then the

¹This abuse of notation making the temperature a function of the GHG concentration corresponds only to the case when the temperature is slaved to the concentration, as given by equation (2.1). Our definition of the equilibrium sensitivity is different from that of the Intergovernmental Panel on Climate Change (IPCC) and is given by the change in steady-state temperature that occurs with a doubling of atmospheric CO₂, $\Delta T(c) = (S_{\text{eq}}/\ln 2) \ln[c/c_0]$, so $\Delta T(2c_0) = S_{\text{eq}}$.

associated Ornstein–Uhlenbeck process can be described using the Ito procedure by a Fokker–Planck (FP) equation,

$$\frac{\partial p(\Delta c, t)}{\partial t} = \frac{\partial}{\partial \Delta c} \left[\left(\frac{\Delta c}{\tau_c} - q(t) \right) p(\Delta c, t) + \frac{\Gamma}{2} \frac{\partial p(\Delta c, t)}{\partial \Delta c} \right],$$

for the time-dependent pdf, $p(\Delta c, t)$ (Cao & Silbey 2008), where $\langle f_c(t_1)f_c(t_2) \rangle = \Gamma \delta(t_1 - t_2)$. For the case of constant emission rate, $q(t) = \text{const.}$, the solution of the above FP equation is (Gardiner 2004)

$$p^c(\Delta c, t | \Delta c(0)) = \frac{1}{\sqrt{\pi \Gamma \beta(t)}} \exp \left\{ -\frac{1}{\Gamma \beta(t)} [(\Delta c - q\tau_c) - (\Delta c(0) - q\tau_c)e^{-t/\tau_c}]^2 \right\}, \quad (2.2)$$

following the initial increment in the concentration of GHGs and $\beta(t) = 2\tau_c(1 - e^{-2t/\tau_c})$. The appropriate boundary conditions for this FP equation are $p(\Delta c(0), t) = p(\infty, t) = 0$. The above expression is of the form $f(\Delta c - \bar{c})$ and vanishes when $\Delta c = \pm\infty$ and may thus be used to write the correctly normalized anti-symmetric combination $(1/2)f(\Delta c - \bar{c}) - f(\Delta c + \bar{c})$ valid over the allowed range.

The time-dependent pdf for the temperature is found by the change of variables $\gamma \Delta c = \Delta T / (\Delta T_0 + \Delta T)$, with $\gamma \neq 0$, which is restricted to the range, $0 \leq \Delta c \leq \gamma^{-1}$ (or equivalently, $0 \leq \Delta T \leq \infty$), and transforms $p^c(\Delta c, t | \Delta c(0))$ according to $p(\Delta T, t | \Delta T(0)) = p^c(\Delta c, t | \Delta c(0)) (d\Delta c/d\Delta T)$ so that

$$p(\Delta T, t | \Delta T(0)) = \frac{1}{\sqrt{\pi \Gamma \beta(t)}} \exp \left\{ -\frac{1}{\Gamma \beta(t) \gamma^2} [Q(\Delta T) - Q(\Delta T(0))e^{-t/\tau_c}]^2 \right\} \\ \times (\Delta T_0 + \Delta T)^{-2} \Delta T_0 |\gamma|^{-1}, \quad (2.3)$$

where

$$Q(\Delta T) = \left(\frac{\Delta T}{\Delta T_0 + \Delta T} \right) + \Delta c(0) - \gamma q \tau_c.$$

In figure 1, we show the temporal evolution associated with this function for parameters chosen to be consistent with current global warming scenarios. At very long times, the stationary pdf for the temperature difference,

$$p_s(\Delta T) = p(\Delta T, \infty | \Delta T(0)) = \frac{1}{\sqrt{2\pi \Gamma \tau_c}} \exp \left\{ -\frac{1}{2\Gamma \gamma^2 \tau_c} [Q(\Delta T)]^2 \right\} \\ \times (\Delta T_0 + \Delta T)^{-2} \Delta T_0 |\gamma|^{-1},$$

exhibits a fat tail in agreement with the results of Roe & Baker (2007) and implies that the expected value $\langle \Delta T(t) \rangle = \infty$ for all times. We note that, here the fat tail in the pdf does not reflect a mechanism giving rise to Levy flights, but is simply the consequence of the nonlinear change of variables in the pdf from concentration to temperature, with an underlying conventional Gaussian random process. For the parameters based on IPCC data and projections (Intergovernmental Panel on Climate Change 2007) (see figure 1 caption for details), the maximum in the pdf in 100 years occurs at $\Delta T = 2.66^\circ\text{C}$ as the concentration increases from 350 to 550 ppm. However, the probability is 0.24 that $\Delta T \geq 4^\circ\text{C}$ and 0.10 that $\Delta T \geq 5^\circ\text{C}$.

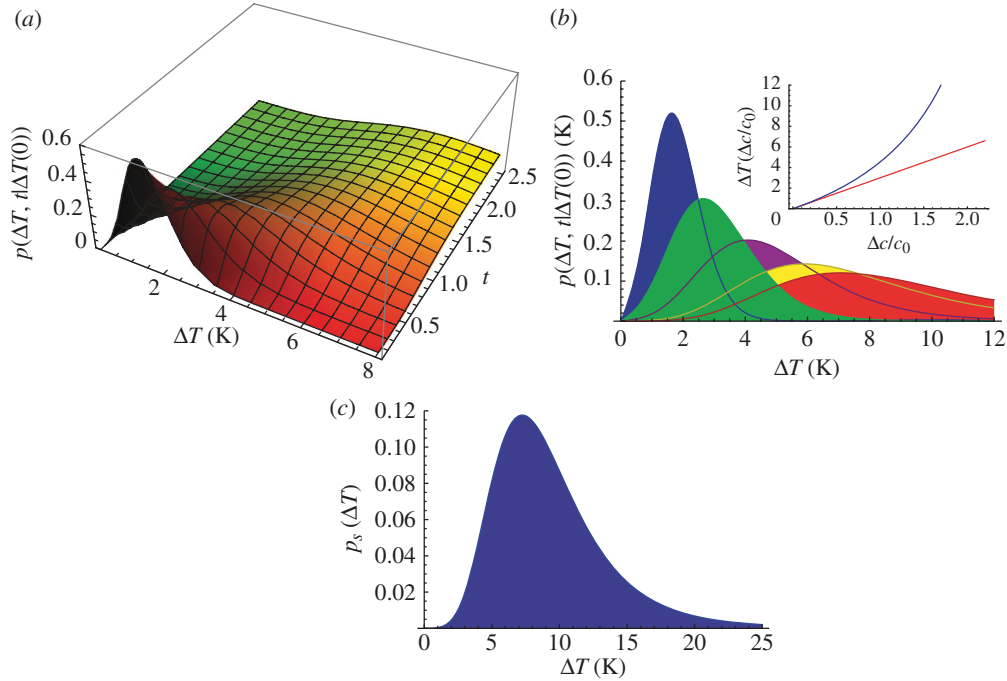


Figure 1. Evolution of the normalized pdf for the case of positive feedback and fast temperature relaxation for $p(\Delta T, t|\Delta T(0))$, equation (2.3). For illustrative purposes, parameters have been chosen to match conventional global warming scenarios (<http://www.ipcc.ch/>). The initial CO₂ concentration is $c(0) = 350$ ppm, $c_0 = 275$ is the pre-industrial value, the concentration relaxation time is $\tau_c = 200$ years, the value $q\tau/\Delta c(0) = 2.45$ is chosen so that, after 100 years, $t/\tau_c = 0.5$ in dimensionless units of time, the concentration reaches the value $c(1) = 550$ ppm, twice the pre-industrial level. The feedback parameter $\gamma = 0.34$ and the equilibrium temperature difference $\Delta T_0 = 8.84 = 3/\gamma$ are chosen so that the most probable temperature increase is $\Delta T(0.5) = 4.54$ K after 100 years. With these choices, the steady-state concentration $c(\infty) = q\tau_c$ is 858 ppm, which gives a most probable temperature increase of $\Delta T(\infty) = 22.1$ K with $\Delta T(0) = 0.91$ K. In all figures, the strength of the random concentration fluctuations is arbitrarily chosen so that $\Gamma = 0.1$ to display a credible temperature evolution. (a) A three-dimensional view of $p(\Delta T, t|\Delta T(0))$; $\Delta T(0) = 2.22$ K. (b) Cross-sectional views of the surface shown in (a) at dimensionless times $t = 0.1$ (blue), 0.5 (green), 1 (purple), 2 (yellow) and (4) red. The inset shows the scaled deterministic change in the equilibrium temperature as a function of the concentration in the limit when temperature is the fast variable, according to equation (2.1) showing how the temperature (in blue) increases with positive feedback in the presence of GHGs; the red line is the linear approximation valid for small changes in GHG concentration. (c) The stationary distribution $p_s(\Delta T) = p(\Delta T, \infty|\Delta T(0))$ for the parameter values given above.

For negative feedback, $\gamma < 0$,² the steady-state condition now covers the entire concentration range, $0 \leq c \leq \infty$, but the temperature is restricted to the range $0 \leq \Delta T \leq \Delta T_0$. The expression for the time-dependent pdf, equation (2.3), remains valid with the modifications $\Delta T/\Delta T_0 = |\gamma|\Delta c/(1 + |\gamma|\Delta c)$ and $Q(\Delta T) = [\Delta T/(\Delta T_0 - \Delta T) - 1 - |\gamma|q\tau_c]$. Beyond the range $\Delta T > \Delta T_0$, $p(\Delta T, t|\Delta T_0)$ is zero and there is no fat-tailed distribution.

²In the IPCC Synthesis Report 2007, fig. 5.1 implies that climate action arises with negative feedback (Intergovernmental Panel on Climate Change 2007).

We next consider the case of slow temperature relaxation, i.e. $\tau_T/\tau_c \gg 1$. This situation arises, for example, when considering the long time scale for energy transfer from the atmosphere to the ocean (Baker & Roe 2009). On the fast time scale, the concentration will adopt its steady-state value according to equation (1.4) and $c_s(t) = q(t)\tau_c$. On this time scale, there is no dynamic feedback and the linearized stochastic equation for the temperature, equation (1.3), is

$$\frac{d\Delta T(t)}{dt} = \frac{\lambda_0 \Delta R_p(c_s(t))}{\tau_T} - \frac{[1 - \gamma \Delta c_s(t)] \Delta T(t)}{\tau_{\text{eff}}} + f_T(t). \quad (2.4)$$

For the case of linear radiative forcing and constant c_s , the stationary temperature, ΔT_e , is shifted higher, according to the relation $\Delta T_e = \Delta T_0 \gamma \Delta c_s / (1 - \gamma \Delta c_s)$. This stochastic equation can again be expressed as an FP equation with a solution given in terms of the temperature difference $\delta T(t) = \Delta T(t) - \Delta T_e$ as

$$p(\delta T, t | \delta T(0)) = \frac{1}{\sqrt{\pi \Gamma_T \hat{\beta}(t)}} \exp \left\{ -\frac{1}{\Gamma_T \hat{\beta}(t)} [\delta T - \delta T(0) \exp(-t/\hat{\tau})]^2 \right\}, \quad (2.5)$$

with $\hat{\beta}(t) = 2\hat{\tau}[1 - \exp(-2t/\hat{\tau})]$ and $\hat{\tau} = \tau/(1 - \gamma \Delta c_s)$.

Although this pdf does not show a fat tail, the system now exhibits critical slowing down since $\langle \Delta T(t) \rangle - \Delta T_e = [\Delta T(0) - \Delta T_e] \exp(-t/\hat{\tau})$ and $\hat{\tau} \rightarrow \infty$ as $(\gamma \Delta c_s) \rightarrow 1$. This corresponds to either a relatively large value of the feedback factor γ or a large concentration of GHG in the atmosphere.

For the case of slow temperature relaxation, it is possible for $p(\delta T, t | \delta T(0))$ to exhibit a fat tail if there is a subsidiary pdf for the feedback parameter γ or equivalently g_0 . For example, if the gain $g = (1 - g_0)(1 - \gamma \Delta c_s)$ is distributed according to a pdf $h(g)$, which is finite at $h(1)$, then, as shown in appendix B, $p(\delta T, t | \delta T(0)) \rightarrow h(1)\delta T^{-2}$ as $\delta T \rightarrow \infty$.

3. Discussion

In summary, we find that, if the temperature relaxes on a fast time scale compared with that for concentration equilibration, the temperature is slaved to the variability in the concentration. For positive feedback, $\gamma > 0$, both stationary and time-dependent pdfs for the temperature change have fat tails owing to the stochastic dynamics of the concentration field. In contrast, if the temperature relaxes on a slow time scale compared with the time scale for concentration equilibration, the stationary and time-dependent pdfs do not possess fat tails, and the stochastic dynamics of the temperature determines the form of the time-dependent pdf. In particular, the characteristic relaxation time exhibits critical slowing down, as the singular point of the positive feedback is approached. However, if uncertainty is introduced in the feedback variable, γ , a fat tail may reappear.

We conclude with an application of our results to a policy issue. Modellers of global climate know that feedback effects are crucial. They observe simulation outcomes that are skewed to high temperatures (Webster *et al.* 2001, 2003; Forest *et al.* 2002). The existence of a fat tail is not an artefact of computational

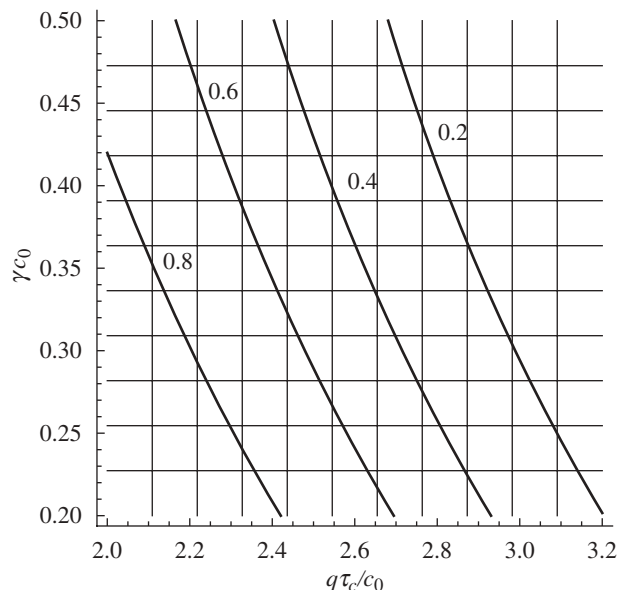


Figure 2. Policy plot. Covariation of the reduced emission rate $q\tau_c/c_0$ and the reduced feedback parameter γc_0 at dimensionless time t/τ_c to keep the cumulative probability of realizing a most probable temperature increase less than some prescribed value ΔT_{\max} . The figure shows contours of the surface $S(q, \gamma)$ corresponding to the cumulative probability distribution $S(q, \gamma) = \int_0^{\Delta T_{\max}} p(\Delta T, t|\Delta T(0))d\Delta T \leq P_{\max}$ given the temperature pdf in equation (2.3) for four values of P_{\max} , with $\Delta T_{\max} = 5$ and the reduced time $t/\tau_c = 1$. Other parameter values are as follows: $\tau_c = 200$ years, $\Gamma = 0.1$, $c_0 = 275$ ppm and $\Delta T_0 = 3/\gamma^\circ\text{C}$.

GCM simulation that will disappear with repeated Monte-Carlo trials, but rather an inherent consequence of the presence of positive feedback, as our analysis of the simple model examined here shows. pdfs with fat tails present formidable problems for conventional expected value cost–benefit analysis because of the relatively higher probability of high cost outcomes (in this case elevated global temperature) (Roe 2009; Weitzman 2009); for example, a simple consequence is that damage functions of the form $D(\delta T) \propto (\delta T)^\alpha$ with $\alpha \geq 1$ will have unbounded expected value at all times. Any practical policy designed to reduce the likelihood or consequence of loss probability outcomes must address not only the shape of the stationary pdf, but also its time evolution. Our model analysis offers an insight into the relaxation times that may be encountered, and the explicit form for the pdf permits an examination of the trade-offs between parameters that might be achieved by policy action. For example, it is possible to determine all combinations of effluent flux, q , and positive feedback, γ , that give the probability, P_{\max} , of a temperature increase below a set amount, ΔT_{\max} . If the cost is known of policy options that are available to lower the emission rate q or reduce the feedback parameter γ (by geo-engineering), then an optimal economic policy can be determined. Figure 2 presents an example of such a trade-off that exhibits a striking and somewhat unexpected linear relation between the combinations of q and γ that results in a given P_{\max} .

Our analysis of the simple climate model can be easily extended to the case in which there are several feedback mechanisms, both positive and negative, acting simultaneously (Roe 2009; Weitzman 2009). This approach can also be applied to a climate that shifts between positive and negative feedbacks by matching solutions for the time-dependent pdfs. In the case of fast temperature relaxation, if the flux q varies with time on the slow time scale, the resulting FP equations can be solved numerically. Extending the treatment to include the spatial variation of the concentration $c(\mathbf{r}, t)$ and temperature $T(\mathbf{r}, t)$ requires including the effects of convection and/or diffusion in the transport equations. As the resulting equations are nonlinear, mode coupling will almost certainly result in a variety of new behaviours. However, the zero-mode behaviour reported here should persist.

Appendix A. Derivation of stochastic dynamical system for climate

We begin with the deterministic energy balance equation for the atmosphere, obtained by integrating across its depth and over all latitudes, equation (1.1),

$$\rho C_p H \frac{dT(t)}{dt} = R(c) - \sigma(1 - g(c)) T(t)^4.$$

In the context of climate change driven by changes in the concentration of GHGs, we are interested in small deviations from the equilibrium temperature T_0 consistent with a (pre-industrial) equilibrium concentration of GHGs, c_0 , defined by

$$R(c_0) = \sigma(1 - g(c_0)) T_0^4. \quad (\text{A } 1)$$

Small deviations from this state, $\Delta T(t) = T(t) - T_0$, lead to

$$\rho C_p H \frac{d\Delta T(t)}{dt} = R_p(c) - 4\sigma T_0^3(1 - g(c)) \Delta T(t),$$

where $R_p(c) = [R(c) - R(c_0) - \sigma \Delta g(c) T_0^4]$, with $\Delta g = g(c) - g_0$ and $g_0 = g(c_0)$. We now expand with respect to $\Delta c(t) = c(t) - c_0$ in two steps. First, expand $g(c)$ to first order and keep the leading coupling term $\Delta T \Delta c$,

$$\rho C_p H \frac{d\Delta T(t)}{dt} = \Delta R_p(c) - \sigma(1 - g_0)(4T_0^3)[1 - \gamma \Delta c] \Delta T(t), \quad (\text{A } 2)$$

where $\Delta R_p(c) = [R(c) - R(c_0) + \sigma g'_0 T_0^4 \Delta c]$ and $\gamma = g'_0/(1 - g_0)$, with $(\cdot)' \equiv d(\cdot)/dc$. The radiative forcing term $\Delta R_p(c)$ can be nonlinear in c . Dividing both sides of equation (A 2) by $(1 - g_0)(\sigma 4T_0^3)$ and defining $\lambda_0^{-1} = (1 - g_0)(\sigma 4T_0^3)$ and $\tau_T = \lambda_0 \rho C_p H$ yields the deterministic form of the incremental energy balance equation,

$$\frac{d\Delta T(t)}{dt} = \frac{\lambda_0 \Delta R_p(c(t))}{\tau_T} - \frac{[1 - \gamma \Delta c]}{\tau_T} \Delta T(t). \quad (\text{A } 3)$$

If we now expand the radiative forcing to the lowest order in Δc , this yields $\Delta R_p(c) = [R'(c_0) - \sigma g'_0 T_0^4] \Delta c$.

This derivation is consistent with the traditional approach that starts with a linearized energy balance equation given by (Tung 2007)

$$\rho C_p H \frac{d\Delta T}{dt} = R(c) - A - B\Delta T, \quad (\text{A } 4)$$

where the right-hand side is the difference between the incoming solar flux Q on Earth with albedo a , $R(c) = (1 - a)Q$, and the outgoing radiation is given by approximating the Stefan–Boltzmann black-body radiation law via a linearized approximation determined by fitting the form of infrared emission from the Earth to observational data on outgoing long-wave radiation (Torn & Hart 2006), with $A = (\sigma T^4)_0$ and $B = (d(\sigma T^4)/dT)$, where the subscript 0 corresponds to the steady state. To make the connection, we again consider small perturbations in the temperature and the GHG concentration from the pre-industrial equilibrium, introducing $\Delta T(t) = T(t) - T_0$ and $\Delta c(t) = c(t) - c_0$, in equation (A 4), and obtain

$$\rho C_p H \frac{d\Delta T(t)}{dt} = [R'(c_0) - B'(c_0)T_0]\Delta c - [B(c_0) + B'(c_0)\Delta c]\Delta T(t), \quad (\text{A } 5)$$

which is of the same form as equation (A 2). Dividing both sides by $B(c_0)$ and letting the scaled incremental radiative forcing $\lambda_0 \Delta R_p = (R'(c_0) - B'(c_0)T_0)/B(c_0)$, the characteristic relaxation time scale for temperature variations $\tau_T = \rho C_p H/B(c_0)$ and $\gamma = -B'(c_0)/B(c_0)$ yields exactly the same deterministic version of the energy balance equation (A 3). Adding the scaled form of a fluctuating term $f_T(t) = \hat{f}_T(t)/(1 - g_0)(\sigma 4 T_0^3)$ that characterizes rapid uncorrelated fluctuations in the radiative forcing yields equation (1.3). The equation for the evolution of GHGs, equation (1.2), is a first-order process in the presence of source and fluctuations.

Appendix B. The role of feedback uncertainty in the case of fast relaxation of GHGs

For fast concentration relaxation, the general form for the average temperature change as a function of time is

$$\tau_T \left\langle \frac{d\Delta T(t)}{dt} \right\rangle = \lambda_0 \Delta R_p(c_s(t)) - [1 - g(c_s(t))]\langle \Delta T(t) \rangle, \quad (\text{B } 1)$$

where $\lambda_0^{-1} = (4\sigma T_0^3)$. This can be formally integrated to yield

$$\begin{aligned} \langle \Delta T(t) \rangle = & \exp \left[-\frac{1}{\tau_T} \int_0^t ds [1 - g(c_s(t))] \right] \Delta T(0) + \exp \left[-\frac{1}{\tau_T} \int_0^t ds [1 - g(c_s(t))] \right] \\ & \times \int_0^t ds \left\{ \exp \left[\frac{1}{\tau_T} \int_0^s du [1 - g(c_s(t))] \right] \lambda_0 \Delta R_p(c_s(t)) \right\}. \end{aligned} \quad (\text{B } 2)$$

The relaxation of $\langle \Delta T(t) \rangle$ slows down as $g(c_s(t)) \rightarrow 1$. If we assume $c_s(t)$ is a constant, equation (B 2) simplifies to

$$\begin{aligned} \langle \Delta T(t) \rangle &= \exp \left[-\frac{t}{\tau_T} [1 - g(c_s)] \right] \Delta T(0) + \frac{\lambda_0 \Delta R_p(c_s)}{[1 - g(c_s)]} \\ &\times \left\{ 1 - \exp \left[-\frac{t}{\tau_T} [1 - g(c_s)] \right] \right\}. \end{aligned} \quad (\text{B } 3)$$

We recall that, to first order $[1 - g(c_s)] = (1 - g_0)[1 - \gamma \Delta c_s]$, so that the slowing down can occur either for $g_0 \rightarrow 1$ or for $\gamma \rightarrow 1$. Different values of the gain, $g = g(c_s)$, will lead to different average temperature changes; thus in equations (B 2) or (B 3), the temperature increase should be denoted by $\langle \Delta T(t) \rangle(g)$, indicating that the average temperature change is a function of g . Different values of $\{g_i\}$ will also produce different trajectories for $\langle \Delta T(t) \rangle(g)$. If one assumes that the different values for g come from a pdf $h(g)$ supported on $0 \leq g \leq 1$, this produces a pdf for $\Delta \hat{T}(t)$ of the form

$$p(\Delta \hat{T}(t)) = h(g^{-1}(\Delta \hat{T}(t))) \frac{dg}{d\Delta \hat{T}(t)}, \quad \Delta \hat{T}_{\min}(t) \leq \Delta \hat{T}(t) \leq \infty, \quad (\text{B } 4)$$

with $\Delta \hat{T}_{\min}(t) = \exp(-t/\tau_T) \Delta T(0) + \lambda_0 \Delta R_p(c_s) \{1 - \exp(-t/\tau_T)\}$. As long as $h(g)$ is finite at $g = 1$, the pdf $p(\Delta \hat{T}(t))$ will be asymmetric and exhibit a fat tail. As an example, consider equation (B 3), in the limit $t \rightarrow \infty$, $g(c_s) \rightarrow 1$, with $\bar{t} = t(1 - g(c_s))$ fixed. In this limit,

$$\langle \Delta T(t) \rangle \rightarrow \frac{\Delta T_s}{[1 - g(c_s)]} \left\{ 1 - \exp \left(-\frac{\bar{t}}{\tau_T} \right) \right\},$$

with $\Delta T_s = \lambda_0 \Delta R_p(c_s)$. This gives

$$\frac{d\langle \Delta T(t) \rangle}{dg(c_s)} = \frac{\langle \Delta T(t) \rangle^2}{\Delta T_s} \left\{ 1 - \exp \left(-\frac{\bar{t}}{\tau_T} \right) \right\}^{-1}$$

and the pdf for the temperature increase,

$$p(\Delta \hat{T}(t)) = h \left[1 - \frac{\Delta T_s [1 - \exp(-\bar{t}/\tau_T)]}{\Delta \hat{T}(\bar{t})} \right] \frac{\Delta T_s [1 - \exp(-\bar{t}/\tau_T)]}{\Delta \hat{T}(\bar{t})}$$

over the range $\Delta T_s [1 - \exp(-\bar{t}/\tau_T)] \leq \Delta \hat{T}(\bar{t}) \leq \infty$, which shows a fat tail.

References

- Baker, M. B. & Roe, G. H. 2009 The shape of things to come: why climate change is so predictable. *J. Clim.* **22**, 4574–4589. (doi:10.1175/2009JCLI2647.1)
- Cao, J. & Silbey, R. J. 2008 Generic schemes for single-molecule kinetics. 1: self-consistent pathway solutions for renewal processes. *J. Phys. Chem. B* **112**, 12 867–12 880. (doi:10.1021/jp803347m)
- Forest, C. E., Stone, P. H., Sokolov, A. P., Allen, M. R. & Webster, M. D. 2002 Quantifying uncertainties in climate system properties with the use of recent climate observations. *Science* **295**, 113–117. (doi:10.1126/science.1064419)
- Gardiner, C. W. 2004 *Handbook of stochastic methods*, ch. 5, 2nd edn. Berlin, Germany: Springer.

- Graves, C. E., Lee, W. H. & North, G. R. 1993 New parameterizations and sensitivities for simple climate models. *J. Geophys. Res.* **98**, 5025–5036. (doi:10.1029/92JD02666)
- Intergovernmental Panel on Climate Change (IPCC). 2007 *Synthesis Report 2007*. See http://www.ipcc.ch/pdf/assessment-report/ar4/syr/ar4_syr.pdf.
- Roe, G. H. 2009 Feedbacks, timescales, and seeing red. *Ann. Rev. Earth Planet. Sci.* **37**, 93–115. (doi:10.1146/annurev.earth.061008.134734)
- Roe, G. H. & Baker, M. B. 2007 Why is climate sensitivity so unpredictable? *Science* **318**, 629–632. (doi:10.1126/science.1144735)
- Scheffer, M., Brovkin, V. & Cox, P. M. 2006 Positive feedback between global warming and atmospheric CO₂ concentration inferred from past climate change. *Geophys. Res. Lett.* **33**, L10 702. (doi:10.1029/2005GL025044)
- Torn, M. S. & Hart, J. 2006 Missing feedbacks, asymmetric uncertainties, and the underestimation of future warming. *Geophys. Res. Lett.* **33**, L10 703. (doi:10.1029/2005GL025540)
- Tung, K. K. 2007 Simple climate modeling. *Discrete Continuous Dyn. Syst. B* **7**, 651.
- Wang, J. & Wolynes, P. 1995 Intermittency of single molecule reaction dynamics in fluctuating environments. *Phys. Rev. Lett.* **74**, 4317–4320. (doi:10.1103/PhysRevLett.74.4317)
- Wang, J. & Wolynes, P. 1999 Intermittency of activated events in single molecules: the reaction–diffusion description. *J. Chem. Phys.* **110**, 4812. (doi:10.1063/1.478369)
- Webster, M. D., Forest, C. E., Riley, J. M., Sokolov, A. P., Stone, P. H., Jacoby, H. D. & Prinn, R. G. 2001 Uncertainty analysis of global climate change projections. Report no. 73, MIT Joint Program on the Science and Policy of Climate Change.
- Webster, M. *et al.* 2003 Uncertainty analysis of climate change and policy response. *Clim. Change* **61**, 295–320. (doi:10.1023/B:CLIM.0000004564.09961.9f)
- Weitzman, M. L. 2009 On modeling and interpreting the economics of catastrophic climate change. *Rev. Econ. Stat.* **91**, 1–19. (doi:10.1162/rest.91.1.1)