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Barrier Penetration and Superluminal Velocity

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We show rigorously in a causal theory that a gaussian wave packet, incident from the left on a potential barrier, turns up on the right side of the barrier at an earlier time than would appear to be allowed by causal propagation. A sufficient set of conditions on the wave packet and barrier parameters for this phenomenon to take place is given. These conditions are quite restrictive but may not all be necessary. There is, of course, only an apparent violation of causality. © 1993 Academic Press, Inc.

I. INTRODUCTION

It has been known for many years that a straightforward application of conventional wave packet theory to barrier penetration predicts that a wave packet takes zero *additional* time[1] to traverse the barrier [1, 2].

In quantum theory, the natural question to ask is whether one can actually *measure* an anomalously high velocity (or short transition time). The answer, given by Mende and Low [3], is that one *cannot*. The key is the word "additional" in the preceding paragraph. That is, it turns out that the total time taken by a wave packet to traverse a distance $L+b$, with b the barrier width, is

$$T = L/V_0, \tag{1}$$

where V_0 is the free space group velocity of the wave packet. However, it is shown in Ref. [3] that for the measurement to make sense, L must be so large that the average velocity,

$$\bar{V} = \frac{L+b}{T} = \left(1 + \frac{b}{L}\right) V_0 \tag{2}$$

is only slightly larger than V_0.

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[1] This does not count small end effects which are independent of the width of the barrier.

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An entirely different situation arises in a relativistic theory, where the free space velocity is the velocity of light (here taken to be 1),

$$V_0 = 1. \tag{3}$$

In that case, it would seem not to be permissible for $\bar{V}$ to be given by Eq. (2) and thus be greater than one. However, we shall here show rigorously that it is permissible for this to happen in a causal model.

There are two realistic situations where this effect can occur. The first consists of a wave guide with cutoff frequency ω_0 into which is inserted a wave guide with a higher cutoff frequency, ω_1. A wave with frequency between ω_1 and ω_0 will see the narrower (higher cutoff frequency) wave guide as a barrier. We do not consider this example further, since the problem of the joining regions of the wave guides does not easily lend itself to rigorously controllable approximation.

The second is the case investigated by Landauer and Martin [2]. An electromagnetic wave is excited in a wave-guide with cutoff frequency ω_0. With dielectric constant ε, the dispersion relation relating propagation constant κ and frequency ω is

$$\varepsilon\omega^2 = \kappa^2 + \omega_0^2. \tag{4}$$

Therefore a wave excited in region I with $\varepsilon > 1$ and therefore with

$$\kappa_1^2 = \varepsilon\omega^2 - \omega_0^2 > 0 \tag{5}$$

would find, on entering region II with $\varepsilon = 1$,

$$\kappa_2^2 = \omega^2 - \omega_0^2. \tag{6}$$

If κ_2^2 is less than zero, then region II is a barrier, and the wave will decay. Landauer and Martin showed with this system that the center of gravity of the wave arrives in a third region (with the same dielectric constant as in region I) as if it took zero time to cross the barrier.

This system is amenable to rigorous analysis, but our goal here is more ambitious than that of the cited authors. We wish to calculate the entire transmitted wave form, not only the center of mass location; we wish to calculate an upper limit to the error in our result; and, we wish to exhibit explicitly the apparent superluminal velocity in a causal theory. All of these goals can, in fact, be carried out for the first system described above. However, the calculations involve some subtle issues and are substantially more complicated than the ones we will carry out in a simple but physically plausible model which we propose below.

Our causal model consists of a classical field $\psi(x, t)$ satisfying a one-dimensional Klein–Gordon equation with variable potential

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi = 0, \tag{7}$$

where

$$V(x)=0 \qquad x<0$$
$$V(x)=m^2, \qquad 0<x<b \tag{8}$$
$$V(x)=0 \qquad x>b.$$

We choose $\psi(x, 0)=f(x)$, a given function, and

$$\frac{\partial\psi}{\partial t}(x, t)\bigg|_{t=0}=g(x). \tag{9}$$

This model has the causal property that if

$$g(x)=0, \qquad x>a,$$

and

$$f(x)=0, \qquad x>a,$$

and $x>a$, then

$$\psi(x, t)=0 \qquad \text{for} \quad t<x-a. \tag{11}$$

That is, the propagation of the wave front is bounded by light velocity.

Similarly, if

$$g(x) \text{ and } f(x)=0, \qquad x<a,$$

and $x<a$, then

$$\psi(x, t)=0 \qquad \text{for} \quad t<a-x. \tag{12}$$

Our actual calculation will be carried out with $f(x)$ and $g(x)$ gaussian centered at a large negative coordinate x_0, and with a mean frequency ω_0:

$$f(x)=e^{i\omega_0 x}\, e^{-((x-x_0)/\Delta x)^2}. \tag{13}$$

We let $g(x)= -\partial f(x)/\partial x$, so that our initial wave packet is moving to the right with velocity $v=c=1$. We shall show that for $x>b$, that is, to the right of the barrier, ψ is substantially proportional to $f(x-x_0-t-b)$,

$$\psi \alpha f(x-x_0-t-b); \tag{14}$$

that is, the wave arrives at x at a time

$$t\approx x-x_0-b \tag{15}$$

which is earlier than the minimum time t_m allowed by light propagation,

$$t_m = x - x_0. \tag{16}$$

However, for this solution to be demonstrably accurate, we shall have to require that the tunneling amplitude be very small, that is, that the exponential

$$e^{-\alpha(\omega_0) b} = e^{-b(m^2 - \omega_0^2)^{1/2}} \ll 1. \tag{17}$$

Here $i\alpha(\omega_0)$ is the (imaginary) wave-number inside the barrier at frequency ω_0. It may be that our conditions are more stringent than necessary; there is some experimental evidence for this [4].

The rest of this paper will be devoted to a mathematical derivation of the result. However, before proceeding to that task, we must comment on the consistency of the apparent superluminal propagation, Eq. (15), with the underlying exact causal propagation, Eqs. (11) and (12). Some insight into the strangeness of this result can be obtained by imagining that the field ϕ also carries two spin components, α_1 and α_2, so that

$$\phi = \alpha_1 \phi_1 + \alpha_2 \phi_2, \tag{18}$$

whereas the scalar ψ is the $\alpha_1 + \alpha_2$ component of ϕ:

$$\psi = (\alpha_1 + \alpha_2, \phi) = \phi_1 + \phi_2. \tag{19}$$

Now suppose that $\phi_1 + \phi_2$ correctly compose the gaussian (13), but that

$$\begin{aligned} \phi_1 &= 0, \qquad x > x_0, \\ \phi_2 &= 0, \qquad x < x_0. \end{aligned} \tag{20}$$

Thus the right half of the original gaussian peak ($x > x_0$) consists only of spin α_2, and the left half ($x < x_0$) of spin α_1.

Clearly, neither ϕ_1 nor ϕ_2 will propagate simply (note the discontinuity of each at $x = x_0$), although $\psi = \phi_1 + \phi_2$ will, as we shall show, propagate simply. If we ask for the spin in the left half of the transmitted peak, but ahead of the light arrival time, i.e.,

$$x_0 + t < x < x_0 + t + b, \tag{21}$$

we will find *only* the ϕ_2 spin component, although the left half of the original wave is given completely by ϕ_1. This is illustrated in Fig. 1.

That is, the entire advanced gaussian, including most of the left half, comes from the forward tail of the original gaussian. The remarkable circumstance is that the left half of the advanced gaussian duplicates so precisely the left half of the original gaussian, although the two halves are not point to point causally related. This behavior is so strange that we have made a special effort to maintain actual mathematical rigor. This results in unavoidable algebraic complications, for which we apologize, in what follows.

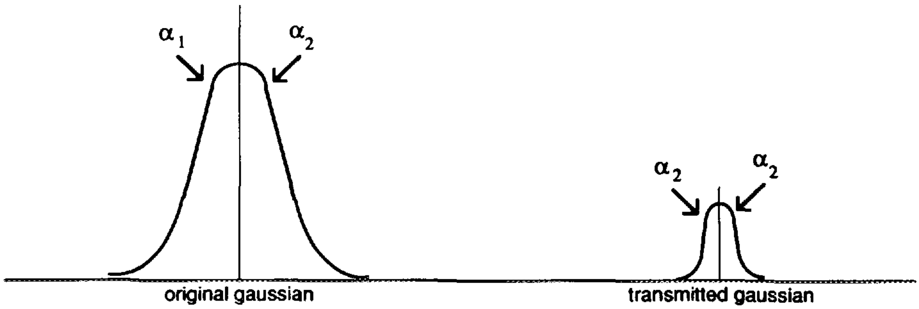

FIGURE 1

II. THE TIME DEPENDENT PROBLEM

We solve the time dependent equation (7) by Laplace transform. With

$$\psi_\omega(x) = \int_0^\infty e^{i\omega t}\,\psi(x, t)\,dt \tag{22}$$

and Im $\omega > 0$, we have, for $t > 0$,

$$\psi(x, t) = \frac{1}{2\pi}\int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} d\omega\, e^{-i\omega t}\,\psi_\omega(x). \tag{23}$$

From Eq. (7) we find

$$\left(-\omega^2 - \frac{\partial^2}{\partial x^2} + m^2\right)\psi_\omega = \frac{\partial \psi}{\partial t}(x, t)\bigg|_{t=0} - i\omega\psi(x, 0) \tag{24}$$

or, for our case, with $(\partial\psi/\partial t)(x, t)|_{t=0} = -\partial f/\partial x$ and $\psi(x, 0) = f(x)$,

$$\left(-\omega^2 - \frac{\partial^2}{\partial x^2} + m^2\right)\psi_\omega = -i\omega f - \frac{\partial f}{\partial x} = F(x). \tag{25}$$

We construct the Green's function for (25) with two solutions of the homogeneous equation

$$\left(-\omega^2 - \frac{\partial^2}{\partial x^2} + m^2\right)\phi_\omega = 0. \tag{26}$$

We choose these for $\phi^{(+)}$ such that

$$\begin{aligned} \phi^{(+)} &= e^{i\omega x} + R(\omega)\,e^{-i\omega x}, && x \leqslant 0,\\ \phi^{(+)} &= Ae^{i\kappa x} + Be^{-i\kappa x}, && 0 \leqslant x \leqslant b,\\ \phi^{(+)} &= Te^{i\omega x}, && x \geqslant b, \end{aligned} \tag{27}$$

and for $\phi^{(-)}$ such that

$$\begin{aligned} \phi^{(-)} &= e^{-i\omega x} + R' e^{i\omega x}, && x \geqslant b, \\ \phi^{(-)} &= A' e^{-i\kappa x} + B' e^{i\kappa x}, && 0 \leqslant x \leqslant b, \\ \phi^{(-)} &= T' e^{-i\omega x}, && x \leqslant 0. \end{aligned} \tag{28}$$

Here $\kappa = (\omega^2 - m^2)^{1/2}$ is chosen to be analytic in the upper half ω plane. This requires, for almost real ω,

$$\begin{aligned} \kappa &= \sqrt{\omega^2 - m^2}, && \omega \geqslant m > 0, \\ \kappa &= i\sqrt{m^2 - \omega^2}, && -m \leqslant \omega \leqslant m, \\ \kappa &= -\sqrt{\omega^2 - m^2}, && \omega \leqslant -m, \end{aligned} \tag{29}$$

where the positive square root is always taken.

A standard calculation requiring continuity of ϕ and its first derivative at the discontinuities of $V(x)$ yields

$$R = \frac{(\omega^2 - \kappa^2)(1 - e^{2i\kappa b})}{D},$$

$$T = \frac{4\omega\kappa}{D} e^{i(\kappa - \omega) b},$$

$$A = \frac{2\omega(\omega + \kappa)}{D},$$

$$B = \frac{-2\omega(\omega - \kappa)\, e^{2i\kappa b}}{D},$$

with

$$\begin{aligned} D &= (\omega + \kappa)^2 - (\omega - \kappa)^2 e^{2i\kappa b}; \\ R' &= e^{-2i\omega b} R, \\ A' &= e^{i(\kappa - \omega) b} A \\ B' &= e^{-i(\omega + \kappa) b} B \\ T' &= T. \end{aligned} \tag{30}$$

We note the upper half plane analyticity of R, T, A, B, D, and A' and of $e^{2i\omega b} R'$ and $e^{2i\omega b} B'$; we also note the absence of upper half plane zeros of D. These conditions are sufficient to guarantee causal propagation from f to ψ. There is a zero of D as $\kappa \to 0$ which we must treat at the appropriate time.

The constant Wronskian $W(\phi^+, \phi^-)$ of $\phi^{(+)}$ and $\phi^{(-)}$,

$$W(\phi^+, \phi^-) = \frac{\partial \phi^{(+)}}{\partial x} \phi^- - \frac{\partial \phi^-}{\partial x} \phi^{(+)}, \tag{31}$$

is

$$W(\phi^+, \phi^-) = 2i\omega T. \tag{32}$$

We can then construct a solution of the inhomogeneous equation (25) as

$$\psi_\omega(x) = -\frac{\phi^+(x)}{2i\omega T} \int_{-\infty}^{x} \phi^-(x')\, F(x')\, dx' + \frac{\phi^-(x)}{2i\omega T} \int_{x}^{\infty} \phi^+(x')\, F(x')\, dx'. \tag{33}$$

Note that, as $x \to \pm\infty$ and ω in the upper half plane, ψ_ω is well behaved. In contrast, no solution of the homogeneous equation is well behaved at both ends, so that (33) is the unique allowable solution of (25).

The solution of the time dependent problem is thus

$$\psi(x, t) = \frac{-1}{2\pi} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} d\omega\, e^{-i\omega t}$$

$$\times \left[\frac{\phi^+(x)}{2i\omega T} \int_{-\infty}^{x} \phi^-(x')\, F(x')\, dx' + \frac{\phi^-(x)}{2i\omega T} \int_{x}^{\infty} \phi^+(x')\, F(x')\, dx' \right]. \tag{34}$$

It is straightforward but tedious to verify, using the analyticity properties given by Eq. (30), that ψ in Eq. (34) has the causality properties of Eq. (11) and (12).

We now wish to consider propagation from the original functions f and g through the barrier to the region with $x > b$. Since $F(x')$ is centered at a point x_0 to the left of zero, it is intuitively clear that for sufficiently large $|x_0|/\Delta x$ we can replace $\phi^-(x')$ (in Eq. (34)) by its value for $x' < 0$ and extend the integral over all x. For the same reason we can drop the second integral in (34). Although this is almost obvious, we insist on actually calculating explicit limits for the neglected terms since we wish to be sure, in light of our surprising result, that there is no possibility of error. These terms are all calculated in the Appendix, where we show that each of them is bounded by $e^{-x_0^2/(\Delta x)^2}$ times a function $mf(mb)$ of m and b. The function f is finite for all finite values of m and b, and hence we can make the correction as small as we wish by taking $|x_0|/\Delta x$ sufficiently large.[2] Thus Eq. (34) can now be written

$$\psi(x, t) = -\frac{1}{2\pi} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \frac{d\omega\, e^{-i\omega t}}{2i\omega T(\omega)} e^{i\omega x}\, T(\omega) \int_{-\infty}^{\infty} T(\omega)\, e^{-i\omega x'} \left(-i\omega f(x') - \frac{\partial f(x')}{\partial x'} \right) dx'. \tag{35}$$

[2] Note that as $|x_0|$ becomes larger, the average velocity $\bar{V} = (x - x_0)/(x - x_0 - b)$ becomes closer to one,

$$\frac{\bar{V} - 1}{\Delta \bar{V}} \cong \frac{b}{x - x_0 - b} \cdot \frac{1}{b\, \Delta x/(x - x_0 - b)^2} \cong \frac{|x_0|}{\Delta x}$$

becomes larger, so that the anomalous behavior is accentuated.

An integration by parts shows that the two terms in (35) contribute equally, leading to a final answer

$$\psi(x, t)=\frac{1}{2\pi}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} e^{i\omega(x-t)}\,d\omega\,T(\omega)\int_{-\infty}^{\infty} e^{-i\omega x'}f(x')\,dx'. \tag{36}$$

It is interesting that, despite all the butchering we have carried out, $\psi(x, t)$ still has two desirable exact properties:

First, for $T=1$, $\psi(x, t)=f(x-t)$, as it should.

Second, and most important, ψ is causally related to f for propagation to the right. Thus, suppose $f(x')=0$, $x>a$. Then (36) becomes

$$\psi(x, t)=\frac{1}{2\pi}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\omega\,T(\omega)\,e^{i\omega(x-t-a)}\int_{-\infty}^{a} e^{-i\omega(x'-a)}f(x')\,dx'. \tag{37}$$

The integral over x' is analytic in the upper half ω plane and goes to zero for $\omega\to\infty$ at least as fast as $1/\omega$; $T(\omega)$ is also analytic in the upper half ω plane and goes to a constant (one) as $\omega\to\infty$. Therefore, for $x-t-a>0$, or $t<x-a$, the ω contour can be closed above, and $\psi=0$. Thus the propagation is causal for any sufficiently convergent f and for *any* T that has upper half plane analyticity.

III. The Simplest Model Solution

We can exhibit the anomalous barrier penetration time with a simplified transmission amplitude

$$T=e^{ib(\kappa-\omega)} \tag{38}$$

and an initial function with $\omega_0=0$,

$$f=e^{-(x-x_0)^2/(\Delta x)^2}. \tag{39}$$

Note that T in Eq. (38) has the required upper half plane analyticity.

We first carry out the integral over x' in (36):

$$\int_{-\infty}^{\infty} dx'\,e^{-i\omega x'}f(x')=\sqrt{\pi}\,\Delta x\,e^{-\omega^2/(\Delta\omega)^2}\,e^{-i\omega x_0}, \tag{40}$$

where

$$\Delta\omega\cdot\Delta x=2. \tag{41}$$

Equation (36) becomes

$$\psi = \frac{\sqrt{\pi}}{2\pi} \Delta x \int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-t-x_0)}\, e^{-\omega^2/(\Delta\omega)^2}\, T(\omega). \tag{42}$$

The usual way of treating Eq. (42) in the quantum theory of scattering is to expand $T(\omega)$ about ω_0 (here zero), and keep only the zeroth term, i.e., set $T(\omega) = T(\omega_0)$. This is valid when

$$\Delta\omega \ll \omega_c$$

or

$$\Delta x \gg L_c = 1/\omega_c, \tag{43}$$

where L_c is the characteristic length of the scatterer; here, $L_c \approx b$. No measurement of penetration time is then possible. In our case, where we explicitly wish to measure a time difference b, we must have $\Delta x < b$, and we must keep more terms in the expansion of $T(\omega_0)$ about ω_0. If we can limit this expansion to a small enough number of terms, we can, at least formally, obtain a closed expression for the transmitted wave for a wide class of functions f falling off rapidly in both x and ω. However, the use of the gaussian for f permits us to give rigorous bounds to the neglected terms in our approximations.

We proceed as follows: we wish to expand $\kappa(\omega)$ (as given by Eq. (29)) near $\omega = 0$ in powers of ω. Clearly, this cannot be valid in Eq. (42) for the entire range of ω. Therefore, we divide the calculation into three parts. First, we require that there exist an m_1 such that the contributions to (42) for $|\omega| > m_1$ are negligible, and such that for $\omega < m_1$ the expansion can stop after the first two terms. Having done that, we go back to the infinite integral by adding back the (still) negligible contributions from values of $|\omega| > m_1$.

The contribution $\delta\psi_>$ from $|\omega| > m_1$ can (since $|T| \leq 1$) be bounded by

$$|\delta\psi_>| < \frac{\Delta x}{\sqrt{\pi}} \int_{m_1}^{\infty} e^{-\omega^2/(\Delta\omega)^2}\, d\omega \tag{44}$$

$$< \frac{2}{\sqrt{\pi}} \frac{\Delta\omega}{m_1} e^{-(m_1/\Delta\omega)^2}. \tag{45}$$

Next we expand $\kappa = i\sqrt{m^2 - \omega^2}$ about $\omega = 0$,

$$\kappa = i\left(m - \frac{\omega^2}{2m} - \frac{\omega^4}{8m^3}\left(1 + \frac{2-1/2}{3}\frac{\omega^2}{m^2} + \frac{(2-1/2)}{3}\frac{(3-1/2)}{4}\frac{\omega^4}{m^4} + \cdots\right)\right) \tag{46}$$

where the parenthesis multiplying $\omega^4/8m^3$ is bounded by $1/(1-\omega^2/m^2)$. Therefore, we can rewrite (42) as

$$\psi = \frac{1}{2\sqrt{\pi}} \Delta x \int_{-m_1}^{m_1} d\omega\, e^{i\omega(x-t-x_0-b)}\, e^{-mb-\omega^2/(\Delta\omega')^2}$$
$$\times \left[\left\{ \left(\exp\left[\frac{b\omega^4}{8m^3} \left(1 + \frac{2-1/2}{3} \frac{\omega^2}{m^2} + \cdots \right) \right] - 1 \right) \Big/ \frac{b\omega^4}{8m^3} \right\} \cdot \frac{b\omega^4}{8m^3} + 1 \right], \tag{47}$$

where

$$\frac{1}{(\Delta\omega')^2} = \frac{1}{(\Delta\omega)^2} - \frac{b}{2m}. \tag{48}$$

Clearly, a condition here is that $1/(\Delta\omega')^2 > 0$.

The "1" term in (47) is almost our final result, that is, a propagated gaussian with width $\Delta\omega'$, but it is incomplete, since it is only integrated between $\pm m_1$. The correction term $\delta\psi_<$ is bounded by

$$|\delta\psi_<| < \frac{3}{64} \Delta\omega'\, \Delta x (\Delta\omega')^4 \frac{b}{m^3} e^{-mb} \left[\exp\left(\frac{bm_1^4}{8m^3} \cdot \frac{1}{1-m_1^2/m^2} \right) - 1 \right] \Big/ \frac{bm_1^4}{8m^3}. \tag{49}$$

This makes $\delta\psi_<$ negligible when the exponent in (49) is of order unity, as we shall see shortly.

Finally, we extend the integral (47) from $\pm m_1$ to $\pm\infty$, this time with an error

$$|\delta\psi'_>| < \frac{1}{\sqrt{\pi}} \frac{\Delta\omega'}{m_1} e^{-mb}\, e^{-m_1^2/(\Delta\omega')^2}. \tag{50}$$

We find for ψ,

$$\psi = \frac{\Delta\omega'}{\Delta\omega} e^{-mb}\, e^{-((x-t-x_0-b)/\Delta x')^2}, \tag{51}$$

where

$$\Delta x'\, \Delta\omega' = 2. \tag{52}$$

Our condition on the validity of the result (51) for ψ must be that all the $d\psi$'s which we have estimated are much smaller than our result (51). To study this question we note first that x_0 can be made as large as we wish without affecting the result (51) or the approximations made for $\delta\psi_>$, $\delta\psi_<$, and $\delta\psi'_>$. Therefore we may simply take $|x_0|$ large enough to give validity (to whatever accuracy we choose) to Eq. (35).

The other conditions are

(i)

(ii)
$$\frac{\Delta\omega}{m_1} e^{-m_1^2/(\Delta\omega)^2} \ll \frac{\Delta\omega'}{\Delta\omega} e^{-mb} \tag{53}$$

(iii)
$$\frac{m_1^4 b}{m^3} \leqslant 1 \tag{54}$$

$$\frac{\Delta\omega'}{m_1} e^{-m_1^2/(\Delta\omega)^2} \ll \frac{\Delta\omega'}{\Delta\omega}. \tag{55}$$

In addition, we want the time displacement b to be significantly outside of Δx, so we must have

(iv)
$$\frac{b}{\Delta x} \cong b\,\Delta\omega \gg 1. \tag{56}$$

Finally, we must verify that m_1/m is not too close to one, and that

$$\frac{1}{(\Delta\omega')^2} = \frac{1}{(\Delta\omega)^2} - \frac{b}{2m} > 0 \tag{57}$$

and is the same order as $(1/\Delta\omega)^2$.

We anticipate the result by ignoring non-exponential factors. Thus, with $m_1/\Delta\omega = z$, $mb = y$ and $m_1/m = u$, we must have

$$\text{from (53),} \quad z^2 \gg y; \tag{58}$$

$$\text{from (54),} \quad u^4 y \leqslant 1; \tag{59}$$

$$\text{and from (56),} \quad \frac{yu}{z} \gg 1. \tag{60}$$

From (60), as $z \to \infty$, so must y (since $u < 1$), and from (59) $u \to 0$. Clearly (57) is automatically satisfied. Furthermore, $(\Delta\omega)^2\, b/m = u^2 y/z^2$, and since $z^2 \gg y$ and $u^2 \ll 1$, we have

$$\frac{(\Delta\omega)^2\, b}{m} \to 0 \qquad \text{and} \qquad \frac{\Delta\omega'}{\Delta\omega} \to 1.$$

The conditions (58)–(60) are compatible; let $y = z^n$, $u = 1/z^p$. Then,

$$\text{from (58),} \quad 2 > n; \tag{61}$$

$$\text{from (59),} \quad n \leqslant 4p; \tag{62}$$

$$\text{and from (60),} \quad n \geqslant 1 + p. \tag{63}$$

Since $4p \geqslant n \geqslant 1+p$ and $n<2$,

$$1 > p \geqslant \tfrac{1}{3} \tag{64}$$

and

$$2 > n \geqslant \tfrac{4}{3}. \tag{65}$$

Thus, our approximations require large $m_1/\Delta\omega$, large mb, and small m_1/m.

To conclude this section, we return briefly to discuss the apparent violation of causal propagation by our solution, Eq. (51). In particular, the amplitude at $x = x_0 + t + b$ must come in the underlying integral (36) from values of x' for which causal propagation to that point is possible, that is,

$$x' > x_0 + b. \tag{66}$$

That means that we can replace the lower limit of the integral over dx' by $x_0 + b$. Equation (36) then becomes

$$\psi(x,t) = \frac{1}{2\pi} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\omega\, T(\omega)\, e^{i\omega(x-t)} \int_{x_0+b}^{\infty} e^{i\omega x'} f(x')\, dx'. \tag{67}$$

This way of writing ψ shows how the front tail of the gaussian gets over the barrier; it jumps over it, since in (67) the integral over x' is like $1/\omega$ as $\omega \to \infty$, *not* like $e^{-\omega^2/\Delta\omega^2}$! In particular, if we consider the contribution ψ_1 of large ω to (67), we set $T=1$ and find, for ψ_1,

$$\psi_1 = \theta(x - t - x_0 - b)\, f(x-t). \tag{68}$$

The first term in (68) is non-zero for all $t < x - (x_0 + b)$, as we would expect; its magnitude $|\psi_1|$ is given by

$$|\psi_1| = e^{-(x-t-x_0)^2/(\Delta x)^2}, \tag{69}$$

and for $x - t \approx x_0 + b$,

$$|\psi_1| \approx e^{-b^2/(\sigma x)^2}. \tag{70}$$

Recall that

$$\frac{b^2}{(\Delta x)^2} = b^2(\Delta\omega)^2 = \frac{y^2u^2}{z^2} \tag{71}$$

which is smaller than $bm = y$, since

$$\frac{yu^2}{z^2} \ll 1. \tag{72}$$

Therefore there is enough strength in the nearby forward field of the gaussian wave-packet to produce the advanced transmission.

IV. The Full Model

We return now to the original barrier problem, with a transmission amplitude given by Eq. (30) as

$$T = \frac{4\omega\kappa\, e^{i(\kappa - \omega) b}}{(\omega + \kappa)^2 - (\omega - \kappa)^2\, e^{2i\kappa b}}. \tag{73}$$

As in Eq. (36), we have

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} d\omega\, e^{i\omega(x - t)}\, T(\omega) \int_{-\infty}^{\infty} e^{i\omega x'} f(x')\, dx'. \tag{74}$$

Our initial amplitude $f(x)$ will now include a mean frequency, ω_0, so that

$$f(x) = e^{i\omega_0 x - (x - x_0)^2/(\Delta x)^2} \tag{75}$$

and

$$\int_{-\infty}^{\infty} e^{-i\omega_0 x'} f(x')\, dx' = \sqrt{\pi}\, \Delta x\, e^{-i\omega x_0 - (\omega - \omega_0)^2/(\Delta\omega)^2 + ix_0\omega_0}. \tag{76}$$

As before, $\Delta\omega\, \Delta x = 2$.

We now proceed as in Section III, noting that $|T| \leqslant 1$ for real ω, so that, as before, we can restrict the ω integration in (74) to the range $-m_1 < \omega < m_1$. The limit on the error so incurred is the same as before, Eq. (45):

$$|\delta\psi_>| < \frac{2}{\sqrt{\pi}} \frac{\Delta\omega}{m_1} e^{-(m_1/\Delta\omega)^2}. \tag{77}$$

We choose m_1 sufficiently small so that we can break off the expansion of $\alpha = \kappa/i$ after the first two terms. Thus, as before,

$$\alpha - \left(m - \frac{\omega^2}{2m}\right) < \frac{m_1^4}{8m^3} \cdot \frac{1}{1 - m_1^2/m^2}. \tag{78}$$

Since $u = m_1/m \ll 1$, and $\omega < m_1$, we can ignore the contribution of $\omega^2/2m$ in (78) except in the exponential. Thus, T becomes

$$T = \frac{4\omega m i}{(im)^2} e^{-mb + \omega^2 b/2m - i\omega b}. \tag{79}$$

plus corrections which can be explicitly calculated and which are bounded by m_1/m. We now have

$$\psi = \frac{\Delta x}{2\sqrt{\pi}} \left(\frac{-4i}{m}\right) \int_{-m_1}^{m_1} \omega \, d\omega \, e^{i\omega(x - t - x_0 - b)} e^{-mb} e^{\omega^2 b/2m - (\omega - \omega_0)^2/(\Delta\omega)^2}. \tag{80}$$

The next step is to rearrange the exponent of (80). The relevant identity is

$$\frac{\omega^2 b}{2m} - \frac{(\omega - \omega_0)^2}{(\Delta\omega)^2} = \frac{\omega_0^2 b}{2m(1 - (\Delta\omega)^2 b/2m)} - \frac{1}{(\Delta\omega')^2}(\omega - \omega_1)^2, \tag{81}$$

where

$$\frac{1}{(\Delta\omega')^2} = \frac{1}{(\Delta\omega)^2} - \frac{b}{2m} \tag{82}$$

and

$$\omega_1 = \omega_0 \left(\frac{\Delta\omega'}{\Delta\omega}\right)^2 = \frac{\omega_0}{1 - (\Delta\omega)^2 b/2m}. \tag{83}$$

We see that

$$\frac{1}{(\Delta\omega')^2} = \frac{1}{(\Delta\omega)^2}\left(1 - \frac{(\Delta\omega)^2 b}{2m}\right)$$

and that

$$\frac{(\Delta\omega)^2 b}{m} = \frac{yu^2}{z^2} \ll 1,$$

so that $\Delta\omega/\Delta\omega' \approx 1$ and $\omega_1/\omega_0 \approx 1$.

Our wave function ψ becomes

$$\psi = \frac{\Delta x}{2\sqrt{\pi}} \int_{-m_1}^{m_1} -\frac{4i\omega}{m} d\omega$$
$$\times \exp\left(i\omega(x - t - x_0 - b) - mb + \frac{\omega_0^2 b}{2m(1 - (\Delta\omega)^2 b/2m)} - \frac{(\omega - \omega_1)^2}{(\Delta\omega')^2}\right). \tag{84}$$

In order to obtain a single closed form for the answer, we extend the integral (84) to $\pm\infty$. The difference is

$$|\delta\psi'_>| < \frac{\Delta x}{\sqrt{\pi}} \int_{m_1}^{\infty} \frac{4\omega}{m} \exp\left\{-mb + \frac{\omega_0^2 b}{2m(1-b(\Delta\omega)^2/m)} - \left(\frac{1}{(\Delta\omega')^2}\right)^{(\omega-\omega_1)^2}\right\} d\omega$$

$$= 4\,\Delta x \frac{(\Delta\omega')^2}{m} e^{-mb(1-\omega_0^2/2m^2)} \left(\frac{1}{2} + \frac{\omega_1}{m_1 - \omega_1}\right) e^{-(m_1-\omega_1)^2/(\Delta\omega)^2}. \qquad (85)$$

In (85) we have neglected the term $(\omega_0^2 b/2m) \cdot ((\Delta\omega)^2 b/2m) = (\omega_0^2/m_1^2) \cdot (y^2u^2/z^2)$ whose value depends on our choice of ω_0^2/m_1^2, which must be less than one, but is otherwise free.

The final expression for ψ is

$$\psi = \frac{4}{m}\left(-i\omega_1 + \frac{x-t-x_0-b}{(\Delta x')^2}\right) e^{-mb(1-\omega_0^2/2m^2) + i\omega_0(x-t-x_0-b)} \cdot e^{-(x-t-x_0-b)^2/(\Delta x')^2} \qquad (86)$$

which is the predicted answer.

APPENDIX: Derivation of Eq. (35)

We start from Eq. (34)

$$\psi(x, t) = -\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{d\omega\, e^{-i\omega t}}{4\pi i\omega T} \left[\phi^+(x) \int_{-\infty}^{x} \phi^-(x')\left(-\frac{\partial f}{\partial x} - i\omega f\right) dx' + \phi^-(x) \int_{x}^{\infty} \phi^+(x')\left(-\frac{\partial f}{\partial x} - i\omega f\right) dx'\right]. \qquad (A.1)$$

The $\partial f/\partial x$ term can be integrated by parts to give

$$\psi(x, t) = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{d\omega\, e^{-i\omega t}}{2\pi T} \left[\phi^+(x) \int_{-\omega}^{x} \frac{f(x')}{2}\left[\phi^-(x') - \frac{1}{i\omega}\frac{\partial\phi^-}{\partial x'}\right] dx' + \phi^-(x) \int_{x}^{\infty} \frac{f(x')}{2}\left[\phi^+(x') - \frac{1}{i\omega}\frac{\partial\phi^+(x')}{\partial x'}\right] dx'\right]. \qquad (A.2)$$

The last term in (A.2) is zero since $x > b$. The first integral must be decomposed into the contributions of the three regions. With $x > b$, this gives

$$\psi(x, t) = \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \frac{d\omega}{2\pi} e^{i(x-t)\omega} \left[T \int_{-\infty}^{0} e^{-i\omega x'} f(x') \right.$$
$$+ \int_0^b dx' f(x') \left[\frac{A'}{2}\left(1 + \frac{\kappa}{\omega}\right) e^{-i\kappa x'} + \frac{B'}{2}\left(1 - \frac{\kappa}{\omega}\right) e^{i\kappa x'} \right]$$
$$\left. + \int_b^x dx' f(x')(e^{-i\omega x'}) \right]. \tag{A.3}$$

We arrive at Eq. (35) by dropping

$$\delta\psi = \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \frac{d\omega}{2\pi} e^{i\omega(x-t)} \left[-T \int_0^{\infty} e^{-i\omega x'} f(x') \right.$$
$$\left. + \int_0^b dx' f(x') \left[\frac{A'}{2}\left(1 + \frac{\kappa}{\omega}\right) e^{-i\kappa x'} + \frac{B'}{2}\left(1 - \frac{\kappa}{\omega}\right) e^{i\kappa x'} \right] + \int_b^x dx' f(x') e^{-i\omega x'} \right]. \tag{A.4}$$

Since $f(x')$ is a gaussian centered at x_0, where x_0 is large and negative, all the integrals on the right go gaussianly to zero as $x_0 \to -\infty$. However, proceeding in this way leads to a subsequent divergent integral over ω. We are thus led to a more complicated method: interchanging the order of integration over ω and x'. In order to do this, we must make two subtractions from the T and A' terms. The B' term is already sufficiently convergent.

We thus write

$$T = T - 1 - i(\kappa - \omega)\, b + 1 + i(\kappa - \omega)\, b. \tag{A.5}$$

To order $1/\omega^2$,

$$\omega - \kappa = \frac{m^2 \omega}{2(m^2 + \omega^2)}, \tag{A.6}$$

where the m^2 in the denominator has been added to provide an absolutely convergent integral over the real axis. Similarly, from Eq. (30),

$$\frac{A'}{2}\left(1 + \frac{\kappa}{\omega}\right) e^{-i\kappa x'} = A \frac{(1 + \kappa/\omega)}{2} e^{-i\omega x'} e^{i(\omega - \kappa)(x' - b)}$$

and

$$A \cdot \frac{(1 + \kappa/\omega)}{2} e^{i(\kappa - \omega)(b - x')}$$
$$= \left[A \cdot \frac{(1 + \kappa/\omega)}{2} e^{i(\kappa - \omega)(b - x')} - 1 + \frac{im^2\omega(b - x')}{2(m^2 + \omega^2)} \right] + 1 - \frac{im^2\omega(b - x')}{2(m^2 + \omega^2)}. \tag{A.7}$$

The two subtractions can be integrated exactly; the remainders are absolutely convergent and can be simply bounded. We first take the "1" terms

$$\delta\psi_1 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(x-t)} \left\{ -\int_0^{\infty} dx'\, e^{-i\omega x'} f(x') + \int_0^b dx' f(x')\, e^{-i\omega x'} + \int_b^x dx'\, e^{-i\omega x'} f(x') \right\} \tag{A.8}$$

or

$$\begin{aligned} \delta\psi_1 &= -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(x-t)} \int_x^{\infty} e^{-i\omega x'} f(x') \\ &= -\theta(x-t-x) f(x-t) \\ &= 0 \qquad \text{for} \quad t>0. \end{aligned} \tag{A.9}$$

Note that the separate integrals contributing to (A.8) are not zero and have various ghost like contributions; for example, the first term in (A.8) gives a contribution $-\Theta(x-t) f(x-t)$, which does not correspond to a physical propagation.

The next subtracted term gives

$$\delta\psi_2 = \frac{1}{2\pi} \int d\omega\, e^{i(x-t)\omega} \left[\frac{im^2 b\omega}{2(m^2+\omega^2)} \int_0^{\infty} dx'\, e^{-i\omega x'} f(x') - \frac{im^2(b-x')\,\omega}{2(m^2+\omega^2)} \int_0^b dx'\, e^{-i\omega x'} f(x') \right]. \tag{A.10}$$

The ω integrals are now sufficiently convergent to exchange integration orders. The ω integral is, with $x-t-x'=y$,

$$\begin{aligned} I(\omega) &= -ib \frac{m^2}{4\pi} \int d\omega\, e^{iy\omega} \frac{\omega}{(m^2+\omega^2)} \\ &= \frac{bm^2}{4} e^{-m|y|} \varepsilon(y). \end{aligned} \tag{A.11}$$

We can now bound $\delta\psi_2$. Since $b-x'<b$,

$$\begin{aligned} \delta\psi_2 &\leqslant \frac{bm^2}{4} \int_0^{\infty} |f(x')|\, dx' \\ &= \frac{bm^2}{4} \frac{(\Delta x)^2}{|x_0|} e^{-(x_0/\Delta x)^2}. \end{aligned} \tag{A.12}$$

We finally turn to the convergent residues in (A.4):

$$\delta\psi_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-t)}\left[-T+1-\frac{im^2\omega b}{2(m^2+\omega^2)}\right]\int_0^{\infty} e^{-i\omega x'} f(x')\,dx'$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-t)}\int_0^b \left\{\left[\frac{A(1+\kappa/\omega)}{2}e^{i(\kappa-\omega)(b-x')}\right.\right.$$

$$\left.\left.-1+\frac{im^2\omega(b-x')}{2(m^2+\omega^2)}\right]e^{-i\omega x'}+\frac{B'(1-\kappa/\omega)}{2}e^{-i\kappa x'}\right\}dx' f(x'). \qquad \text{(A.13)}$$

The first integral in (A.13), $\delta\psi_{3.1}$, is bounded by

$$|\delta\psi_{31}| < \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\left|-T+1-\frac{im^2\omega b}{2(m^2+\omega^2)}\right| \times \frac{(\Delta x)^2}{|x_0|}e^{-(x_0/\Delta x)^2}. \qquad \text{(A.14)}$$

The integral over ω is a finite function of m and b; we call it $mf(mb)$, where

$$f(0)=0$$

and

$$f(x)=\frac{x\log x}{2\pi}+O(x) \qquad \text{as} \quad x\to\infty. \qquad \text{(A.15)}$$

The second integral has two subtle points. First, the A and B terms cannot be separated, since they both have a singularity at $\kappa=0$ which cancel against each other. Second, the integration parameter x' appears explicitly (and not as a phase) in the integral. We proceed as follows:

$$|\delta\psi_{32}| < \int_0^b dx'\, e^{-(x'-x_0)^2/(\Delta x)^2}$$

$$\times\left\{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|\left[\frac{A(1+\kappa/\omega)}{2}e^{i(\kappa-\omega)(b-x')}-1\right.\right.\right.$$

$$\left.\left.\left.+\frac{im^2\omega(b-x')}{2(m^2+\omega^2)}\right]e^{-i\omega x'}+\frac{B'(1-\kappa/\omega)}{2}e^{i\kappa x'}\right|\right\}. \qquad \text{(A.16)}$$

The integral over ω is a function of m, b, and x', with $0\leqslant x'\leqslant b$. We maximize this function over x', to obtain a second function of m and b, $mg(mb)$. The bound on $\delta\psi_{32}$ is then given by

$$|\delta\psi_{32}| < mg(mb)\,\frac{(\Delta x)^2}{|x_0|}e^{-(x_0/\Delta x)^2}. \qquad \text{(A.17)}$$

The function $g(mb)$ has the values

$$g(0)=0$$

and

$$g(x)=\frac{x \log x}{2\pi}+O(x) \qquad \text{as} \quad x \to \infty. \tag{A.18}$$

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