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Dielectric model of roton interactions in superfluid helium

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The interactions between rotons in superfluid helium are similar to those between dipoles in a polar fluid. Models developed to describe polar fluids may therefore be used to study the influence of roton interactions on various physical quantities in helium II. In this paper we use the Onsager model of a dielectric to calculate the temperature dependence of the energy of the roton minimum in the excitation spectrum, the entropy density, and the normal fluid density, at three different pressures. The results obtained agree well with experimental values for temperatures up to a few tenths of a degree below the λ temperature; the agreement becomes better with increasing pressure. Our theory contains only a single adjustable parameter, the roton hard-sphere radius, for which we find a value close to half the distance between nearest neighbors in the helium fluid.

I. INTRODUCTION

The distinctive properties of liquid helium II were explained by Landau[1] on the basis of a postulated spectrum of elementary excitations, which is shown in Fig. 1. This form of the excitation spectrum was confirmed by microscopic calculations of Feynman[2] and experimentally by inelastic neutron scattering experiments.[3] Thermodynamically the most important regions of the spectrum are the "phonons" at low momentum and the "rotons" near the energy minimum. The latter provide the dominant contributions to all thermodynamic quantities above 1 °K.

In Landau's theory the phonons and rotons are treated as noninteracting bosons. Interactions between quasiparticles, especially between rotons, must be assumed in order to explain the viscosity of helium II and the existence of two-roton bound states.[4] They also manifest themselves in certain aspects of the thermodynamic properties and of the neutron scattering data. For example, the roton minimum in the excitation spectrum, as measured by neutron scattering,[3,5,6] decreases with temperature; the fact that the decrease is, at low T, roughly proportional to the total number of rotons suggests strongly that this shift is due to roton interactions. When the excitation spectrum, as determined from neutron scattering, is used to calculate the entropy or the normal-fluid density, using the independent-particle picture, one finds values which are significantly higher than the experimentally determined ones, especially at higher pressures.

As early analysis of these phenomena was given by Cohen.[7] He developed a quantum-mechanical many-body formalism with which one could, in principle, treat a system of rotons with arbitrary interactions. However, his formalism has not been developed to a point where a comparison with experimental results is possible. In this paper we develop a much simpler treatment, both from a conceptual and from a calculational point of view; however, our method is only applicable for one particular approximate form of the roton-roton interaction.

From Feynman's picture of a roton as a microscopic vortex ring it follows that the superfluid flow pattern at some distance from the roton has a dipolar form. Two rotons at a large relative distance interact mainly via this velocity field. In Sec. II we will show that this interaction has the same form, apart from an over-all minus sign, as that between two electric dipoles in a polar fluid. This analogy was noted by Feynman[2] and further developed by Donnelly and Roberts.[8] In recent years the theory of polar fluids has developed rapidly.[9] The results from that field can be applied to the roton problem when one assumes that the asymptotic form of the roton-roton interaction is exact for all distances larger than some hard-core diameter $2a$; the hard-core repulsion replaces the actual short-distance interaction. Some additional simplifications will be made for calculational convenience.

In Sec. II we will state the analogy between rotons and polar dielectrics and specify our model of roton interactions in more detail. In Sec. III we describe a self-consistent scheme which enables us to calculate the energy of the roton minimum and the roton-number density for various temperatures and pressures. We also show how the normal fluid density and the entropy density may be calculated from these two basic quantities. The entire procedure contains only one adjustable parameter, the hard-sphere radius a, which we choose to be 1.98 Å at 0 atm.

In Sec. IV we present the numerical results for 0, 10, and 20 atm. and compare them with experimental data. In Sec. V we discuss the successes and limitations of the model and some possible refinements. We also compare our results with those obtained by other authors.

By pursuing the analogy between superfluids and dielectrics we are able to construct a theory of liquid helium II in which interactions between rotons can be treated and their effect on various measurable quantities can be evaluated. In the simple form presented here the model gives good agreement with experiment for three physical quantities as a function of temperature: the energy of the roton minimum, the normal-fluid density, and the entropy density. The agreement remains good for temperatures up to a few tenths of a degree from the λ transition, and it improves with increasing pressure. The remaining disagreements can be understood qualitatively and related to the simplifying assumptions made in our treatment.

II. ANALOGY BETWEEN DIELECTRICS AND SUPERFLUIDS

The analogy between dielectrics and superfluids has been developed in detail by Donnelly and Roberts.[8] We briefly restate two basic observations (for simplicity we restrict ourselves to the situation in which the normal fluid is at rest). First, the energy of a roton of momentum $\vec{p}$ changes by an amount $\vec{v}_s \cdot \vec{p}$, when it is placed in a superfluid velocity field $\vec{v}_s$[10]:

$$E_{\vec{v}_s}(\vec{p}) = E_0(\vec{p}) + \vec{p} \cdot \vec{v}_s \,. \tag{2.1}$$

Secondly, around a stationary roton with momentum $\vec{p}$ at $\vec{r}\,'$ there is a superfluid velocity field,[2] which at large distances becomes equal to

$$\vec{v}_s(\vec{r}) = (4\pi\rho)^{-1} \overleftrightarrow{T}(\vec{r} - \vec{r}\,') \cdot \vec{p} \,, \tag{2.2}$$

where ρ is the mass density of the fluid and $\overleftrightarrow{T}(\vec{r} - \vec{r}\,')$ is the dipole tensor

$$\overleftrightarrow{T}(\vec{r} - \vec{r}\,') = -\vec{\nabla}_{\vec{r}} \vec{\nabla}_{\vec{r}\,'} |\vec{r} - \vec{r}|^{-1} \,. \tag{2.3}$$

The relations (2.1) and (2.2) resemble the expressions for the energy of a dipole in an electric field and for the electric field around a dipole; the quantities corresponding to the electric field $\vec{E}$ and the electric dipole moment $\vec{\mu}$ are given by

$$\begin{aligned} \vec{E} &\leftarrow (4\pi\rho)^{1/2} \vec{v}_s \,, \\ \vec{\mu} &\leftarrow (4\pi\rho)^{-1/2} \vec{p} \,. \end{aligned} \tag{2.4}$$

There are two differences between the dielectic and the superfluid case: (i) The sign of the interaction term in Eq. (2.1) is positive; the rotons tend to align *antiparallel* to the superfluid velocity field. (ii) Equation (2.2) is strictly valid only for rotons with velocity zero, i.e., with energies equal to the minimum energy $k\Delta$. This limitation is not too serious, since most rotons have energies quite near to $k\Delta$. We will briefly return to this point in Sec. V.

Various physical quantities in a superfluid have a simple dielectric analog. As an example we mention the normal-fluid density ρ_n. This quantity is related to the total momentum $\vec{P}_{ex}$ carried by all elementary excitations when the superfluid has a small velocity $\vec{v}_s$ with respect to the normal fluid[10]:

$$\vec{P}_{ex} = -\rho_n \vec{v}_s \,. \tag{2.5}$$

A comparison with the expression for the total electric polarization $\vec{P}_{el}$ in an external field $\vec{E}$ in terms of the dielectric constant ϵ,

$$\vec{P}_{el} = \frac{\epsilon - 1}{4\pi} \vec{E} \,, \tag{2.6}$$

gives the correspondence

$$\epsilon \leftarrow 1 - \rho_n / \rho \,. \tag{2.7}$$

Note that the "dielectric constant" of the superfluid is a number between zero and unity, as expected for a substance in which the dipoles tend to align antiparallel to the field (like the induced dipoles in a diamagnetic material).

In order to make further progress we now assume that rotons behave as hard spheres with radius a and interact only via the velocity field $\vec{v}_s$ described by Eq. (2.2). The potential energy for N_r rotons with positions $\vec{r}_i$ and momenta $\vec{p}_i$ is

$$U = \tfrac{1}{2}(4\pi\rho)^{-1} \sum_{i \neq j}^{N_r} \vec{p}_i \circ \overleftrightarrow{T}_{ij}(\vec{r}_i - \vec{r}_j) \cdot \vec{p}_j \,. \tag{2.8}$$

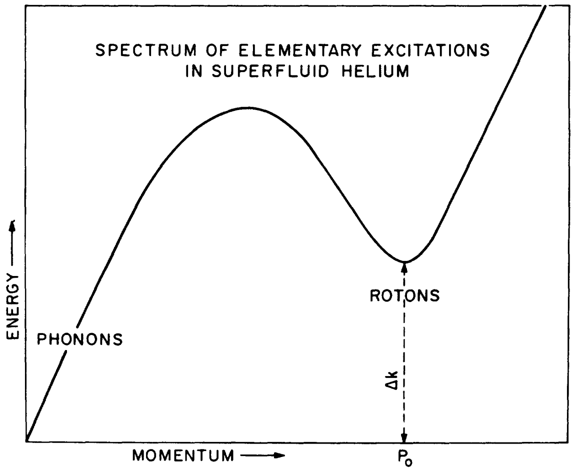

FIG. 1. Spectrum of elementary excitations in liquid helium II as postulated by Landau (Ref. 1). Above 1 °K the most important contributions to the thermodynamics are made by the rotons with momentum near p_0 and energy near $k\Delta$.

As a further simplification we replace the magnitude of each roton's momentum by the magnitude at the roton minimum, i.e., $\vec{p}_i$ is approximated by $\vec{p}_i(p_0/|\vec{p}_i|)$. Then the expression (2.8) is equal, apart from an over-all minus sign, to that of a system of hard spheres with embedded dipole moments of strength $\mu = p_0(4\pi\rho)^{-1/2}$. The replacement of p by p_0 is reasonable in view of the fact that for most rotons p is approximately equal to p_0. If further accuracy is desired, it would be necessary to include corrections in Eq. (2.2) as well, since this equation is accurate only for $p = p_0$.

The specific model for a polar fluid, a system of hard spheres with embedded permanent dipole moments, has received considerable attention recently.[9] Even this simple model cannot be solved exactly, although some Monte Carlo results are available.[11] However, there are a number of approximate theories: the Onsager model,[12] as modified by Nienhuis and Deutch,[13] the mean spherical model, solved by Wertheim,[14] and a thermodynamic perturbation scheme augmented by Padé approximations, which was developed by Stell, Rasaiah, and Narang.[15] In this paper we will use the Onsager model, which is the simplest and adequate for our purposes. (Results with the other two models are briefly discussed in Sec. V.)

A basic quantity in the Onsager model of a dielectric is the quantity

$$y = 4\pi\mu^2 n_d / 9kT\,, \tag{2.9}$$

where n_d is the number of dipoles per cm^3. The corresponding quantity for the roton case is [cf. Eq. (2.4)]

$$y = -p_0^2 n_r / 9kT\rho\,, \tag{2.10}$$

where n_r is the number of rotons per cm^3; the minus sign is connected with the change in sign in the interaction. In terms of the parameter y the dielectric constant of the Onsager model is given by

$$\epsilon(y) = \tfrac{1}{4}\{1 + 9y + [(1+9y)^2 + 8]^{1/2}\}\,. \tag{2.11}$$

This expression leads, via the correspondence (2.7), to an expression for the normal-fluid density in terms of the number of rotons. This expression cannot be trusted at low temperatures, where the contribution of the phonons is important, or at temperatures near the λ point, where excitations with momenta far from the roton minimum contribute significantly. Note that our Eq. (2.7) reduces to the familiar one for the roton contribution to the normal-fluid density,[10]

$$\rho_{nr} = p_0^2 n_r / 3kT\,, \tag{2.12}$$

when we substitute the low-y approximation $\epsilon(y) = 1 + 3y$. The difference between Eqs. (2.11) and (2.12) is caused by the backflow fields of the other rotons. This leads to a local superfluid velocity different from the externally imposed one (cf. the difference between external and local fields in a dielectric).

The ensemble average of the potential energy (2.8) in the Onsager model is[13]

$$\langle U\rangle = \frac{p_0^2 N_r}{32\pi\rho a^3}\,\frac{\epsilon - 1}{2\epsilon + 1}\,. \tag{2.13}$$

The dipolar excess entropy (i.e., the entropy of hard spheres with dipoles minus the entropy of the pure hard-sphere system) is equal to $N_r\,\Delta s_{\mathrm{dip}}$ with[16]

$$\Delta s_{\mathrm{dip}} = \frac{-k}{32\pi n_r a^3}\left(9y\,\frac{\epsilon-1}{2\epsilon+1} + \frac{(\epsilon-1)^2}{\epsilon} + 3\ln\epsilon - \tfrac{9}{2}\ln[\tfrac{1}{3}(2\epsilon+1)]\right)\,. \tag{2.14}$$

These expressions will be used in Sec. III.

III. SELF-CONSISTENT METHOD TO DETERMINE THE ENERGY OF THE ROTON MINIMUM

The results of Sec. II enable one to determine various thermodynamic properties of liquid helium II in terms of the number of rotons. Another quantity which may be determined in this way is the energy of the roton minimum in the elementary excitation spectrum obtained from inelastic neutron scattering. Neutron scattering data give the energy associated with the creation or absorption of one additional roton with momentum $\vec{p}$. This energy is the sum of the energy $E_0(\vec{p})$ needed to create an isolated roton and the energy of the interactions of the new roton with *all* the existing ones. In particular we find for the average energy $k\Delta(\rho, T)$ needed to create one additional roton with momentum p_0

$$k\Delta(\rho, T) = k\Delta(\rho, 0) + 2N_r^{-1}\langle U\rangle\,, \tag{3.1}$$

where $\langle U\rangle$ is given by Eq. (2.13). The energy needed to create an isolated roton is equal to $k\Delta(\rho, 0)$ since the roton density approaches zero when the temperature goes to zero.

Relation (3.1) expresses $\Delta(\rho, T)$ in terms of the number of rotons. To determine both quantities we need a second relation between the same two quantities. This second relation is found by an adaptation of the treatment by Bendt, Cowan, and Yarnell (BCY)[17] of a roton system with temperature-dependent energy levels. These authors neglect the influence of the interactions on the number of quantum levels corresponding to each energy interval. In this approximation they obtain for the number of rotons in a volume V with momenta between p and $p + dp$,

$$N_r(p, dp) = \frac{p^2 V}{2\pi^2\hbar^3}\left[\exp\left(\frac{E(p;\rho, T)}{kT}\right)+1\right]^{-1} dp\,. \tag{3.2}$$

In this expression $E(p;\rho, T)$ is the energy needed to add or remove a single roton with momentum p at helium density ρ and temperature T. The reader is referred to Ref. 17 for the derivation of Eq. (3.2), which is nontrivial for the case of temperature-dependent energy levels.

Equation (3.2) cannot be applied without modification to our model of interacting rotons, since it allows configurations in which the roton hard spheres overlap. This effect is taken into account in an approximate way by replacing the volume V in Eq. (3.2) by an effective volume V^*. The same procedure is used to derive the van der Waals equation of state for a fluid.[18] We determine V^* by comparing the free energy A_{hs} of N hard spheres in a volume V with the ideal-gas value A_{id} for N particles in the same volume at the same temperature. In terms of these quantities V^* is defined by

$$A_{hs}(N, V, T) - A_{id}(N, V, T) = NkT\ \ln(V^*/V)\,. \tag{3.3}$$

We calculate A_{hs} by integrating the pressure of the hard-sphere system over volume from a value V_0 where A_{hs} and A_{id} are virtually identical to the actual volume V:

$$A_{hs}(N, V, T) = A_{id}(N, V_0, T) - \int_{V_0}^{V} p(N/V', T)dV'\,. \tag{3.4}$$

A good approximation to the pressure is given by the Carnahan-Starling hard-sphere equation of state[19]:

$$p(N/V, T) = \frac{3kT}{4\pi a^3}\,\frac{\eta+\eta^2+\eta^3-\eta^4}{(1-\eta)^3} \tag{3.5}$$

where $\eta = \frac{4}{3}\pi a^3 N/V$ is the volume fraction occupied by the hard spheres. By combining Eqs. (3.3)–(3.5) we obtain for the effective volume V^*,

$$V^* \equiv \alpha(n_r)V = V\exp[-(4\eta - 3\eta^2)/(1-\eta)^2]\,. \tag{3.6}$$

For low η this reduces to the familiar result $V^* = V(1-4\eta)$.

We now return to Eq. (3.2). Clearly we must replace V in this equation by V^*. In addition we need an expression for $E(p;\rho, T)$, since Eq. (3.1) gives the value of this function only for $p=p_0$. In view of the approximations made thus far we can expect good results only when almost all rotons have momenta near the roton minimum. We will therefore not lose much accuracy when we replace the actual roton spectrum by a parabola of the form

$$\tilde{E}(p;\rho, T) = k\Delta(\rho, T) + [p - p_0(\rho)]^2/2\mu(\rho, T)\,. \tag{3.7}$$

The assumption that $p_0(\rho)$ is independent of T [as was also assumed in the discussion preceding Eq. (3.1)] is confirmed by neutron scattering experiments.[6] The same experiments indicate[20] that $\mu(\rho, T)$ obeys the relation

$$\mu(\rho, T) = \frac{\Delta(\rho, T)}{\Delta(\rho, 0)}\,\mu(\rho, 0)\,. \tag{3.8}$$

As a last simplification we replace the Planck factor $[\exp(E/kT)+1]^{-1}$ in Eq. (3.2) by the Boltzmann factor $\exp(-E/kT)$. This is justifiable since E is always large compared to kT in the roton region. In this way we obtain the following expression for the total number of rotons per cm^3:

$$\begin{aligned} n_r(\rho, T) &= \frac{\alpha(n_r)}{2\pi^2\hbar^3}\int_0^\infty p^2 \exp\left(\frac{-\tilde{E}(p;\rho, T)}{kT}\right)dp \\ &\simeq \frac{\alpha(n_r)[p_0(\rho)]^2}{2\pi^2\hbar^3}\left(\frac{2\pi kT\Delta(\rho, 0)}{\mu(\rho, 0)\Delta(\rho, T)}\right)^{1/2} \\ &\quad\times \exp\left(\frac{-\Delta(\rho, T)}{T}\right)\,. \end{aligned} \tag{3.9}$$

In evaluating the integral we neglected a factor

$$1 + \mu(\rho, T)kT/p_0^2\,,$$

which changes $n_r(\rho, T)$ by less than half a percent.

From Eqs. (3.9) and (3.1), with substitution of Eqs. (2.13), (2.11), and (2.9), we can now determine $n_r(\rho, T)$ and $\Delta(\rho, T)$ in a self-consistent way. (A similar procedure was used by Ruvalds in Ref. 21). For our calculation we need the parameters $\Delta(\rho, 0)$, $\mu(\rho, 0)$, and $p_0(\rho)$, which are taken from experiment, and the hard-sphere radius a, which plays the role of an adjustable parameter. The best value for a is very close to half the average distance between a helium atom and its nearest neighbor; we therefore assume that the hard-sphere radius varies with the inverse cube root of the helium-fluid density. This is exactly the variation with density found[6] for the other length associated with a roton, $\hbar p_0^{-1}$. With this assumption for the density-dependence of a there is only a single free parameter left in our model. Once we know n_r, we can use Eqs. (2.7), (2.11), and (2.9) to determine the normal-fluid density.

Another interesting quantity is the roton contribution to the entropy of 1 cm^3 of helium. From the independent-particle picture and the approximate expression (3.7) for the roton spectrum one obtains for this quantity[10]

$$S_{r,\,id} = kn_r[\Delta(\rho, T)T^{-1} + \tfrac{3}{2}]\,. \tag{3.10}$$

To this result we add the hard-sphere correction,

$$\Delta S_{r,hs} = kn_r \ln[\alpha(n_r)], \tag{3.11}$$

with $\alpha(n_r)$ defined by Eq. (3.6), and the dipolar correction

$$\Delta S_{r,\mathrm{dip}} = n_r \Delta s_{\mathrm{dip}}, \tag{3.12}$$

with Δs_{dip} given by Eq. (2.14). The contributions (3.11) and (3.12) are both negative, as expected. When we compare the result with experiment, we must also add the phonon contribution, which is equal to[10]

$$S_{\mathrm{ph}} = \frac{16\pi^5 k}{45}\left(\frac{kT}{2\pi\hbar c(\rho, T)}\right)^3, \tag{3.13}$$

where $c(\rho, T)$ is the velocity of sound. Since the phonon contribution itself is rather small we make no attempt to correct this "ideal-gas" value. Our prediction for the entropy per cm^3 is therefore equal to

$$S = S_{\mathrm{ph}} + S_{r,id} + \Delta S_{r,hs} + \Delta S_{r,\mathrm{dip}}; \tag{3.14}$$

the terms are defined in Eqs. (3.10)–(3.13).

The model developed in this section and Sec. II predicts the energy of the roton minimum, the normal-fluid density, and the entropy density as a function of temperature for various densities in terms of a number of known experimental quantities associated with zero temperature and a single adjustable parameter, the roton hard-sphere radius at a single density. In the course of the derivation we made a number of approximations, to which we will return in Sec. V. In Sec. IV we will present the numerical results of our calculations and compare them with the values found in various experiments.

IV. RESULTS AND COMPARISON WITH EXPERIMENTS

The procedures described in the preceding sections give values for the three physical quantities Δ, ρ_n, and S as a function of temperature for various densities, given the parameters of the roton spectrum at zero temperature at the same densities and a value for the roton hard-sphere radius. The experimental data are given as a function of temperature and pressure. For a comparison we need the equation of state for liquid helium II; as such we use the empirical equation of state obtained by Brooks and Donnelly[22] from a numerical analysis of various experimental data.

The values of the parameters entering our calculations are given in Table I for three densities corresponding to 0, 10, and 20 atm at $T = 0$. The momentum at the roton minimum p_0 and the hard-sphere radius a are strictly proportional to the cube root and the inverse cube root of the density, respectively. The values of $\Delta(\rho, 0)$ and $\mu(\rho, 0)$ at densities encountered for higher temperatures at the three pressures considered are obtained by linear interpolation in ρ. The 25-atm data used for this purpose are given in the last column of Table I.

Table II lists our results for the energy of the roton minimum $\Delta(p, T)$, the entropy density S/ρ, and the normal-fluid ratio at 0, 10, and 20 atm for temperatures between 1 °K and the λ point, together with experimental values for the latter two quantities. Experimental values given for $p = 0$ are actually obtained at saturated vapor pressure; even at the λ point this pressure is only 0.05 atm. The experimental values for S/ρ are the smoothed values given by van den Meijdenberg *et al.*[23]; the data at 0 atm are extrapolated to 2.1 °K using the specific heat measurement of Lounasmaa and Kojo.[24] Experimental values for ρ_n/ρ at $p = 0$ are those of Tough *et al.*[25] At higher pressures the high-temperature data are direct measurements by Romer and Duffy.[26] The low-temperature values for ρ_n/ρ are obtained from the velocity of second sound, as measured by Maurer and Herlin,[27] via the relation

$$u_2^2 = \frac{\rho - \rho_n}{\rho_n} \frac{T(S/\rho)^2}{(C_p/\rho)}, \tag{4.1}$$

where u_2 is the velocity of second sound and C_p is specific heat per cm^3 at constant pressure. In determining ρ_n from u_2 and Eq. (4.1) we employed values of S/ρ from Ref. 23, the values of C_V measured by Wiebes and Kramers,[28] and the correction factor C_p/C_V, which is close to unity, from model calculations by Brooks and Donnelly.[22] The indirectly determined values of ρ_n/ρ agree well with the directly measured ones for temperatures where both are available (see Ref. 26).

Experimental values of $\Delta(p, T)$ are only available

TABLE I. Values for several physical parameters in helium II as used in our calculation. The values for the density ρ, the energy and momentum at the roton minimum, Δ and p_0, the roton effective mass μ, and the roton hard-sphere radius a are those at $T = 0$ for various pressures. The numbers are discussed, e.g., in Ref. 22 (except a, which is a parameter specific to our theory). The values for the transition temperature T_λ are given for illustrative purposes only; our theory becomes unreliable for temperatures too close to T_λ.

p (atm)	$\rho(0)$ (g/cm^3)	$\Delta(0)$ (°K)	$\mu(0)$ (m_{He})	$p_0(0)$ (10^{-19} g cm/sec)	$a(0)$ (Å)	T_λ (°K)
0	0.1451	8.65	0.160	2.01	1.98	2.18
10	0.1589	8.02	0.145	2.07	1.92	2.06
20	0.1685	7.47	0.134	2.11	1.88	1.92
25	0.1725	7.20	0.130	···	···	···

TABLE II. Values for the energy of the roton minimum Δ, roton density n_r, entropy density S/ρ, and normal-fluid fraction ρ_n/ρ, as a function of temperature, as calculated from our theory at three different pressures. Experimental values for S/ρ are taken from Ref. 23; those for ρ_n/ρ are from Ref. 25 for $p = 0$ and from Ref. 26 at 10 and 20 atm (the 10-atm data are obtained by interpolation from data at 10.37 and 5.28 atm). Numbers marked with an asterisk are obtained by combining data from several experiments; the procedure is described in Sec. IV. The values of the density ρ for different temperatures and pressures are taken from Ref. 22.

p (atm)	T (°K)	ρ (g/cm^3)	Δ (°K)	n_r (cm^{-3})	$(S/\rho)_{\text{th}}$ (J/g °K)	$(S/\rho)_{\text{expt}}$ (J/g °K)	$(\rho_n/\rho)_{\text{th}}$	$(\rho_n/\rho)_{\text{expt}}$
1	1.0	0.1451	8.64	9.43×10^{18}	0.016	···	0.006	···
1	1.1	0.1451	8.63	2.19×10^{19}	0.029	···	0.013	0.015
1	1.2	0.1451	8.61	4.45×10^{19}	0.049	0.0515	0.025	0.028
1	1.3	0.1451	8.58	8.18×10^{19}	0.078	0.0865	0.042	0.047
1	1.4	0.1451	8.53	1.39×10^{20}	0.119	0.1325	0.066	0.074
1	1.5	0.1452	8.47	2.22×10^{20}	0.174	0.196	0.097	0.113
1	1.6	0.1452	8.39	3.36×10^{20}	0.242	0.284	0.135	0.170
1	1.7	0.1453	8.28	4.90×10^{20}	0.328	0.398	0.181	0.238
1	1.8	0.1454	8.16	6.84×10^{20}	0.426	0.545	0.233	0.325
1	1.9	0.1455	8.01	9.25×10^{20}	0.538	0.732	0.290	0.430
1	2.0	0.1456	7.84	1.21×10^{21}	0.658	0.963	0.348	0.556
1	2.1	0.1458	7.66	1.54×10^{21}	0.785	1.226*	0.404	0.727
10	1.0	0.1589	8.00	1.81×10^{19}	0.018	···	0.012	···
10	1.1	0.1590	7.98	4.01×10^{19}	0.035	···	0.024	···
10	1.2	0.1591	7.93	7.87×10^{19}	0.061	0.0635	0.042	0.041*
10	1.3	0.1592	7.88	1.40×10^{20}	0.099	0.102	0.069	0.066*
10	1.4	0.1593	7.79	2.35×10^{20}	0.151	0.157	0.106	0.102*
10	1.5	0.1595	7.67	3.72×10^{20}	0.221	0.232	0.153	0.151*
10	1.6	0.1597	7.53	5.62×10^{20}	0.308	0.332	0.211	0.217*
10	1.7	0.1601	7.32	8.24×10^{20}	0.416	0.464	0.281	0.312
10	1.8	0.1605	7.07	1.17×10^{21}	0.540	0.637	0.359	0.418
10	1.9	0.1611	6.75	1.61×10^{21}	0.674	0.863	0.440	0.564
10	2.0	0.1618	6.40	2.14×10^{21}	0.807	1.164	0.516	0.768
20	1.0	0.1686	7.43	3.18×10^{19}	0.025	···	0.020	···
20	1.1	0.1687	7.38	6.81×10^{19}	0.048	···	0.039	···
20	1.2	0.1688	7.31	1.30×10^{20}	0.084	0.086	0.068	0.066*
20	1.3	0.1690	7.21	2.30×10^{20}	0.135	0.135	0.109	0.101*
20	1.4	0.1692	7.06	3.84×10^{20}	0.207	0.205	0.165	0.155*
20	1.5	0.1695	6.85	6.15×10^{20}	0.302	0.300	0.238	0.227*
20	1.6	0.1700	6.55	9.60×10^{20}	0.426	0.430	0.330	0.335
20	1.7	0.1705	6.13	1.47×10^{21}	0.577	0.602	0.440	0.457
20	1.8	0.1713	5.62	2.17×10^{21}	0.733	0.838	0.549	0.610

for a few temperatures. The measured values of Dietrich *et al.*[6] and of Henshaw and Woods[5] are presented in Fig. 2; the smooth curves are drawn through our calculated values. The lowest-pressure data in Ref. 6 were obtained at 1 atm.

The agreement between our model calculations and the experimental data is good up to a few tenths of a degree from the λ point, and it improves with increasing pressure. In particular we succeeded in obtaining good agreement for both the roton minimum and the entropy density at high pressures and not too close to T_λ. Such agreement could not be obtained by Dietrich *et al.*[6] or by Brooks and Donnelly[22] by means of a treatment in which roton interactions are neglected. The calculated entropy values at 0 atm differ rather strongly from the experimental ones, even for temperatures far below T_λ. This discrepancy may be attributed completely to our approximation of the roton spectrum by a parabola. The effect of this approximation may be judged by means of a comparison with the work of Bendt *et al.*,[17] who calculated the entropy using the actual excitation spectrum, but neglecting roton interactions. From Fig. 2 of Ref. 17 the contributions of various parts of the spectrum may be determined separately.

To give an impression of the order of magnitude of the various contributions to the entropy density we consider the situation at $T = 1.8$ °K. The roton interactions result in a reduction of the independent quasiparticle expression for the entropy by 9%, 18%, and 34% at 0, 10, and 20 atm respectively. Inclusion of excitations other than phonons and rotons will increase the entropy with approximately 20%, 10%, and 5% at the same pressures. This agrees with the general picture: independent quasiparticle calculations are too high at high temperature and the disagreement gets worse with increasing pressure[17,6]; our values are too low and the disagreement becomes less with increasing pressure (see also the remarks on linewidth effects in Sec. V). In Table III we give both the total value of S/ρ from Ref. 17 and the contributions from the phonons and the region near the roton minimum; the contributions from these two regions will be predicted correctly in our treatment. One sees from Table III that the difference between our calculated values of S/ρ and the experimental ones is, at least at low temperatures, essentially equal to the contribution of excitations with momenta outside of the phonon and roton regions determined by Bendt *et al.* The excellent agreement obtained by Bendt *et al.*[17] at high temperatures is slightly fortuitous. Subsequent experiments[5,6] showed that their values of $\Delta(T)$ are somewhat too high at high temperatures.

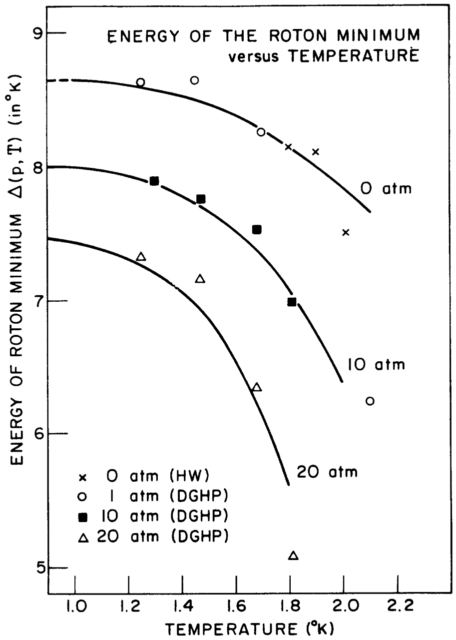

FIG. 2. Theoretical and experimental values for the energy of the roton minimum as a function of temperature at 0, 10, and 20 atm. The experimental points are taken from Dietrich *et al.* (DGHP, Ref. 6) and Henshaw and Woods (HW, Ref. 5). The theoretical curves are drawn through the values given in column 4 of Table II.

V. DISCUSSION

A. Comparison with other treatments

The model presented in this paper provides a simple physical picture of roton interactions and allows us to evaluate the effect of those interactions on three physical quantities: the shift in the roton minimum, the normal-fluid density, and the entropy density. Earlier papers were more limited in scope. Bendt *et al.*[17] and Dietrich *et al.*[6] calculated the entropy density from the experimental excitation spectrum; Brooks and Donnelly[22] took the form of the excitation spectrum from experiment, but determined the energy of the roton minimum from the observed values of the entropy density. The theory used in each of these papers neglects the effects of roton interactions; as a consequence their results are rather poor at high pressures, where interaction effects are important. At low pressure their results are superior to ours, since they use the observed form of the excitation spectrum, rather than its approximation by a parabola.

A treatment somewhat similar to ours was given by Ruvalds.[21] He argued that the shift in the roton minimum is in lowest-order preturbation theory equal to

$$k[\Delta(T)-\Delta(0)]=-2g_4 n_r(T)\,; \tag{5.1}$$

TABLE III. Experimental values of the entropy density S/ρ as a function of T (°K) at 0 atm (in $J/\mathrm{g}\,^\circ\mathrm{K}$) from Ref. 23 and theoretical values obtained by us, compared with the values calculated by Bendt *et al.*[17] The last column, OM, gives the theoretical values from the Onsager model. The third, $(\mathrm{BCY})_{\mathrm{tot}}$, and fourth, $(\mathrm{BCY})_{pr}$, columns give, respectively, the contribution from all elementary excitations and the contribution from the roton and phonon regions only. The numbers suggest that, except at temperatures close to T_λ, the discrepancy between our values and the experimental ones is caused entirely by our use of an approximate excitation spectrum, in which only the phonon and roton regions are described correctly. For the further discussion see the last part of Sec. IV.

T	Expt.	$(\mathrm{BCY})_{\mathrm{tot}}$	$(\mathrm{BCY})_{pr}$	OM
1.2	0.0515	0.051	0.048	0.049
1.3	0.0855	0.084	0.077	0.078
1.4	0.1325	0.132	0.114	0.119
1.5	0.196	0.196	0.168	0.173
1.6	0.284	0.283	0.237	0.242
1.7	0.398	0.398	0.328	0.328
1.8	0.545	0.545	0.441	0.426
1.9	0.732	0.738	0.586	0.538
2.0	0.963	0.983	0.761	0.658
2.1	1.266	1.315	0.986	0.785

the coupling parameter g_4 is treated as an adjustable parameter. He employed Eq. (5.1) and the free-particle expression for the roton density,

$$n_r(T)=\frac{p_0^2}{2\pi^2\hbar^3}(2\pi\mu kT)^{1/2}\exp\left(\frac{-\Delta(T)}{T}\right), \qquad (5.2)$$

to determine $n_r(T)$ and $\Delta(T)$ self-consistently. The normal-fluid density is determined from the free-particle expression (2.12). A good fit for the experimental values of ρ_n at 0 atm is obtained for $g_4 = 3\times10^{-38}$ erg/cm^3; a good fit to $\Delta(T)$ at 1 atm requires $g_4=5\times10^{-38}$ erg/cm^3. At higher pressures it is even harder to obtain good values for ρ_n and Δ simultaneously. This is not too surprising; in Eq. (5.2) and (2.12) the roton interactions, which become more important at high pressure, are neglected.

When the temperature is increased one reaches a point where the Eqs. (5.1) and (5.2) no longer have solutions. Contrary to the impression given in Fig. 2 of Ref. 21, this happens for a finite value of Δ; at 0 atm the value of ρ_n at the endpoint is slightly higher than ρ. The solution of our model, on the other hand, can be continued to temperatures above the λ point; in this region $\Delta(T)$ becomes almost constant. The experimental values of $\Delta(T)$ also reach a constant value,[6] but that value is significantly lower than the one predicted by our theory.

The behavior of Ruvalds's model for increasing T was noticed earlier by Jäckle.[29] He identifies the point at which Eqs. (5.1) and (5.2) no longer have solutions with the λ transition; the normal-fluid fraction at this point is made equal to unity by means of an *ad hoc* modification of the free-particle relation between roton density and normal-fluid density. The singularity in the specific heat, however, is not the finite jump expected for a second-order phase transition in the mean-field approximation, but a $(T-T_\lambda)^{-1/2}$ divergence.

Nagai[30] and Tüttö[31] developed theories analogous to Ruvalds's in which the roton linewidth as well as the shift in $\Delta(T)$ are considered. It turns out to be very difficult to find values for the various coupling constants which give good values for both these quantities when only two-roton terms in the perturbation series are considered.

An expression of the form (5.1) was derived by Kebukawa[32] by means of a perturbation treatment of Bogoliubov's theory of quantum liquids.[33] Straightforward perturbation theory would lead to divergences, but these are eliminated by a renormalization procedure,[34] in which the quasiparticle energies of Bogoliubov are replaced by the ones obtained by Feynman and Cohen.[2] When one assumes that the interaction between the renormalized quasiparticles is the same as that between the unrenormalized ones, one obtains Eq. (5.1) with $g_4 \simeq 18.10^{-38}$ erg/cm^3. This value is much too large to give good agreement with experiment. The magnitude of the discrepancy was not noticed by Kebukawa, since he calculated the roton number from the zero-temperature excitation spectrum. The need for a renormalization of the coupling constant in Kebukawa's theory was demonstrated recently by Rajagopal *et al.*[35]

The theories of Ruvalds and Kebukawa are based on an expansion in terms of the roton-roton coupling parameter, and consequently the shift in Δ is proportional to n_r at low T. In "dielectric" theories the natural expansion variable is $y\sim n_r/T$ and our shift of Δ with temperature is proportional to n_r/T rather than n_r. Unfortunately the linear range is too small and the experimental values are too uncertain to determine which of the two relations is closer to reality. The replacement of a straightforward density expansion by a low-density-high-temperature expansion is a typical feature of systems with long-range forces.[36] In the roton system this "hot, dilute" limit is reached for T going to zero; the decrease in the roton density with decreasing T is much stronger than the increase of the factor T^{-1}. The anomaly is caused by the fact that temperature and density are no longer independent variables in the roton system.

We conclude with a number of remarks on the approximations made in this paper and on possible ways to improve the theory.

B. Roton-roton interactions

The long-range part of the interaction we used is generally believed to be correct for rotons with momentum near p_0.[2,4] For other rotons Feynman's conservation of current argument[2] would predict a dipolar flow pattern proportional to p-vm, where v is the roton velocity and m the mass of a helium atom. Further corrections, especially for rotons with momenta larger than p_0, may be required because of hybridization between the one- and two-roton branches of the excitation spectrum, as predicted, e.g., by Zawadowski, Ruvalds, and Solana.[37] One could allow a variation of the equivalent dipole moment of a roton with momentum and develop an analogy with the theory of polar-fluid mixtures[16] rather than that of simple polar fluids.

The replacement of the short-range roton interaction by a hard-sphere repulsion is a drastic simplification and its consequences are difficult to estimate. The distance of closest approach for two rotons used in this paper, 3.96 Å at zero pressure, is physically reasonable; it is almost exactly equal to the nearest-neighbor distance between two helium atoms (4.015 Å for an fcc configuration at the same density). This agrees well with the intuitive impression that two rotons cannot be centered on the same atom.

A similar estimate of the cutoff in the dipole-dipole interactions follows from the treatment by Miller, Pines, and Nozières,[38] who explain the dipolar interaction as an exchange of a phonon. This suggests deviations from the dipolar form when the exchanged momentum approaches the roton region. This corresponds to distances of the order of the nearest-neighbor distance.[2]

Estimates of the roton hard-sphere radius from the viscosity of helium[4] lead to a much smaller value for the hard-sphere radius (around 1 Å). On the other hand[4] a calculation of the energy of the two-roton bound state leads to a cutoff distance for the dipole-dipole interaction of approximately 5 Å (corresponding to a radius of 2.5 Å). However, the latter calculation uses a potential which vanishes below 5 Å. When a hard-core repulsion is introduced in the formalism of Ref. 4, the estimate for a will surely decrease.

From the examples just mentioned one sees that our interaction potential is too simple to describe all aspects of short-range roton-interactions in a satisfactory manner. The general question of scattering and binding of rotons is discussed in Refs. 4 and 23 and in recent papers by Solana *et al.*[39] and by Roberts and Donnelly.[40]

It is perhaps worthwhile to point out that the effect of a hard-core repulsion on the temperature dependence of Δ differs radically from the effect expected from a soft repulsive part in the interaction. The quantity $\langle U\rangle$ in Eq. (3.1) may be written more fully as

$$\langle U\rangle=\int\cdots\int U e^{-U/kT}\Big/\int\cdots\int e^{-U/kT}, \qquad (5.3)$$

in which $\int\cdots\int$ indicates integration over the positions and orientations of all rotons. Configurations in which any two hard spheres overlap do not contribute to the integrals, and the integrand in the numerator is negative semidefinite after integration over orientations. A soft repulsive part in U would cause the integrand to be of indeterminate sign; in some circumstances one might even find an increase of Δ with temperature, as was predicted, e.g., by Parry and ter Haar.[41]

C. Excitation spectrum and linewidth effects

By comparing the actual excitation spectrum in Fig. 1 with the parabola we used as an approximation one sees that our predicted values for the roton density will be systematically too low. However, the decrease of $\Delta(\rho, 0)$ with pressure leads to a decrease of the relative contribution from all momentum regions except the one near the roton minimum. The increase of the velocity of sound (238, 303, and 348 m/sec at 0, 10, and 20 atm, respectively) leads to a decrease in importance of the phonon region and the region to the right of the roton minimum; in the latter region the curve $E(p)$ changes from quadratic to linear at the point where its slope becomes equal to the sound velocity.[3] The contribution from the region near the maximum in $E(p)$ also decreases in importance due to the increase of the value of this maximum (13.8, 14.7, and 14.8 °K, respectively).

Another reason why our roton densities are too low at high temperatures is our failure to account for effects of the width of the excitations; near the λ transition this width is considerable, and the creation and absorption peaks in the inelastic neutron scattering spectrum actually coalesce at T_λ for momentum transfer near the roton minimum.[6] One might take the finite lifetime into account by using a Boltzmann factor averaged over the line profile rather than one associated with the maximum of the line; this was done in Ref. 6.

Alternatively, one might try to include a probability distribution for the possible values of the potential energy of a roton, instead of simply the average value; however, this would require a much more complex theory of dielectrics, which does not yet exist. We expect that a refinement of our calculation, using the actual roton spectrum and correcting for the finite width of the roton levels, would lead to improved agreement with experiments. In particular the higher roton density would lead to a much faster decrease of $\Delta(T)$ when T approaches T_λ, and to higher values of S and ρ_n in this temperature region. Some additional assumptions concerning the temperature dependence of various parameters of the spectrum would however be necessary in such a treatment.

D. Alternative dielectric models

Instead of the Onsager model of a dielectric one may use either the mean spherical model[14] or the thermodynamic perturbation scheme of Stell and co-workers.[15] Calculations using these two other models give practically the same results as those using the Onsager model, at least for temperature values for which the latter agrees well with experiment. For temperatures near the λ point the theory of Stell *et al.* gives somewhat better agreement with experiment than the Onsager theory, but the improvement is not very significant. For example, at 2 °K and 10 atm, the respective values of $\Delta(T)$, S/ρ, and ρ_n/ρ with the thermodynamic perturbation theory are 6.01 °K, 0.854 J/g °K, 56.1%; with the Onsager model, the values are 6.40 °K, 0.807 J/g °K, 51.6%, while the experimental values are 4.42 °K, 1.164 J/g °K, 76.8%.

In the two alternative models the equations corresponding to Eqs. (3.1) and (3.9) may have more than one solution, and the system shows a first-order phase transition from a state with intermediate to one with very high normal-fluid density. In the latter state the value of $\Delta(T)$ lies below 1 °K and the value of the transition temperature depends on the validity of our approximations for completely unphysical values of $\Delta(T)$ and n_r; its physical meaning is therefore rather questionable. We point out in this connection that the λ transition in our model does not correspond to any special point in the analogous dielectric system; the transition is reached when the difference between ρ and our parameter ρ_n, which contains only a roton contribution, becomes equal to the contribution to the actual ρ_n by elementary excitations other than rotons. Since the latter are not included in our treatment, we cannot predict the λ temperature from our model with any accuracy.

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