Surface growth

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1 Introduction

What do we mean by surface growth? Between any two medias, one can find an interface, and this will be the surface we consider. If one media is growing at the expense of the other, then the surface is said to grow, and this is the process we will be interested in.

The two media in question can be of many different kinds. The fluid-fluid interface is probably the easiest to describe. Since particles can move very easily, intermolecular interactions will dominate almost every thing else and the result will be a surface tension force which will tend to smooth the surface. We will however focus on surfaces where the balance of forces is not so obvious. For example at a solid-solid interface, atoms feel the attraction force of their neighbours, but this attraction decays very quickly with distance. The interface can therefore be potentially much more rough, and hard to describe. Depending on the origin of the growth of the interface, very different interfaces can be generated. We will focus here on two types of growth. The first one is random deposition. Molecules coming randomly deposit on the interface, therefore making it grow. This is of importance in the growth of crystals, accumulation of snow flakes and especially molecular beam epitaxy. This last procedure simply consists in adding atoms one by one to a surface to produce micro, or even nano structures. The second type of growth is one that comes from within the interface. The particles inside one media want to expand, which forces the surface to grow. The applications of this type of growth are numerous in biology, be it to describe the evolution of a colony of bacteria or the growth of various tissues. This process also describes dielectric breakdowns and viscous fingering between fluids if the growth if vigorous enough.

In order to describe these processes, many models have been suggested, and we will review some of them briefly here. In most cases, it is difficult to do much analytically to predict the statistics of the interface. However, a large number of simulation have been performed a many numerical results are known. We are also suggesting a model for bacterial growth, a generalization of existing models, and studying some of its properties.

2 Fractal dimension

The concept of fractal dimension is fundamental in the study of surface growth. To define the fractal dimension of an object, following [BS95], we introduce $V(l)$, which is a measure of the volume of an object which is computed by covering the object by volume elements of size l^d, where d is the dimension of the space in which the object lies. We will write $V(l) = N(l)l^d$, where $N(l)$ is the number of volume elements needed to cover the object. We define the fractal dimension of an abject

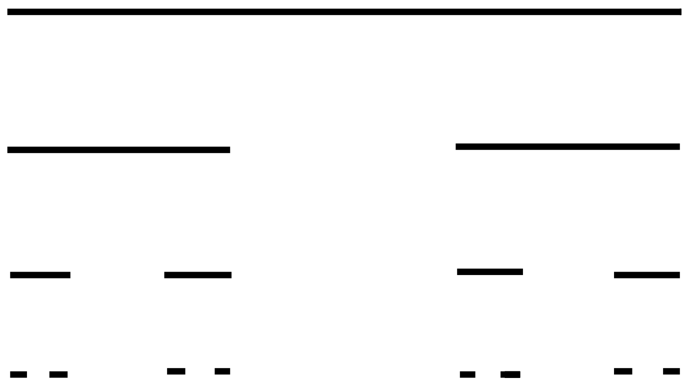


Figure 1: The Cantor set

to be the exponent d_f such that $N(l) = l^{d_f}$, in the limit $l \to 0$. Solving for d_f yields

$$d_f = \lim_{l \to 0} \frac{\log N(l)}{\log 1/l}$$

If we consider for example the Cantor set, at the kth step, the natural length scale to use is $l = (1/3)^k$. Also it is easy to see that $N(l) = 2^k$ so we find $d_f = \log 2/\log 3 = 0.639$. Notice that the fractal dimension here is smaller than one, which means that the object has zero volume. However, d_f is not zero either, so the Cantor set is "bigger" than a point. Notice also that this object, like fractals in general, is self-similar. By that we mean that it has no length natural length scale. Hence looking at the object, one is unable to tell the scale and zooming in does not change the appearance of the object.

More generally, one considers self-affine objects. Once again, there is a rescaling such that the object will appear to be the same, except that this time it will not be isotropic. That is to say that one dimension has to be rescaled differently than the other. If we consider a self-affine curve $h(x)$, it will then obey

$$b^\alpha h(x) = h(bx)$$

To compute the fractal dimension of such a curve, we first notice that, if $l = |x_1 - x_2|$, then

$$\Delta(l) = |h(x_2) - h(x_1)| \approx l^\alpha.$$

Therefore, if we want to cover the whole curve with blocks of size l, we will need $1/l$ along the $x-$axis, and $l^\alpha/l = l^{\alpha-1}$ blocks per column. We then have that

$N(l) = l^{\alpha-2}$ so $d_f = 2 - \alpha$. For example in the case of Brownian motion

$$P(y,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}.$$

So $|y(t_1) - y(t_2)| \approx |t_1 - t_2|^{1/2}$ on average. We thus have $\alpha = 1/2$ and $d_f = 3/2$.

3 Scaling exponents of a surface

We now consider some important properties of a growing surface. The typical case of interest is one where the influx of particles is known and constant, but random. We would now like to characterize the roughness of the surface. If we denote the position of the interface by $h(x,t)$, where x might be a vector, we can define the width of the interface to be the standard deviation of the height

$$w(t) = \sqrt{< h(x,t)h(x',t) >}$$

The size of the system we are considering will also be a determinant factor for the width in most cases so we will generally write $w(L,t)$. Since we consider finite systems, the boundary conditions will be important. We will consider in most cases a periodic boundary condition.

Initially, the surface is supposed to be flat so the width is identically 0. However as time passes, the width grows. It usually increases like a power of t so we define the first exponent β to be such that $w(L,t) \approx t^\beta$ at small times.

If there are correlations between the growth of different sites, the size of the system will matter. We usually consider only positive correlations, meaning that if some region grows fast, it will tend to entrain its surroundings. In that case, and because of the periodic boundary condition, after a long enough time a point will be correlated with the whole system. We therefore expect the system to saturate and the width to remain effectively constant at large times. The size of the system however will greatly influence the final width and so we define the second scaling exponent α to be such that $w(L,t) \approx L^\alpha$ at large times.

We have been talking of "large" and "small" times, but there is usually a rather sudden transition between both regimes, which allows us to introduce a saturation time $\bar{t}$. Of course, the system size will be crucial. This is why we introduce our third scaling exponent z such that $\bar{t} \approx L^z$. Notice that the small and large time regimes have to match at $\bar{t}$. We therefore have

$$\begin{aligned} w(L,t) &= \bar{t}^\beta \\ w(L,t) &= L^\alpha \qquad \text{so,} \\ \bar{t} &= L^{\alpha/\beta} \\ z &= \frac{\alpha}{\beta}. \end{aligned}$$

Let us consider the simplest possible example: random deposition (RD). This is the case of an interface to which a new point is added randomly, with uniform probability. For the one dimensional case, we split the axis into L sites. We then pick a column uniformly, each column with probability $p = 1/L$. We also discretize time such that $t = n/L$ where n is the total number of particles deposited. The time thus define corresponds to the average height. The probability that the height is h is then simply

$$Prob(h, x, t) = \binom{n}{h} p^h (1-p)^{n-h}$$

The variance of the binomial is well-known and it $np(1-p)$ which translates into $w^2(t) = np(1-p) = n/l(1-1/l) \approx t$. Therefore we must have $\beta = 1/2$. notice that there is no saturation time since the growth is not correlated. The other two exponents are therefore not defined.

This case is simple enough to be handled exactly. However, we need to develop a more general approach for more complicated processes. We wish to write a continuous equation describing the growth of the interface. To do so, we consider a constant flux of particles F, and a noise term $\eta(x,t)$, which will now have zero mean. The noise is also uncorrelated, as will usually be the case, so that $< \eta(x,t)\eta(x',t') >= 2D\delta^d(x-x')\delta(t-t')$. In this case, the noise is constant in time and uniform in space. The growth of the interface is then describe by

$$\frac{\partial h}{\partial t} = F + \eta(x,t)$$

In general, the stochastic equation is not so easy to find, but it is the best hope one has to do something analytically.

4 Symmetry Principles

4.1 Linear systems

Let us consider a similar model but with a slight modification. In addition to simply choosing uniformly a column i, we then decide to add a point to the lowest column $i-1$, i or $i+1$, or in higher dimensions, to the lowest neighbouring point. This introduces correlations in the system since the growth probability of a column depends on the height on its neighbours. This problem is very difficult to understand in its discrete version so we seek a continuous equation describing the phenomena.

The way we look for a partial differential equation is basically by using Occam's razor and eliminating terms that contradict symmetries of the system. We start with the ansatz

$$\frac{\partial h}{\partial t} = G(h, x, t) + \eta(x, t)$$

where the noise has the same properties as before and $G(h, x, t)$ is a differential operator to specify. The first thing we can notice about this system is that the growth process is independent of time, i.e. $G(h, x, t) = G(h, x, t') = G(h, x)$. Similarly, the growth process is independent of position. This is only true because of the choice of our boundary condition, which eliminates a dependency on the distance to the boundary. So we are left with $G(h, x) = G(h)$. Continuing along the same lines, we now notice that the growth process is independent of the average height. It will depend on variations of the height, but will not involve h explicitly. So we can assume that $G(h)$ is a differential operator only.

We now want to use the symmetries in the system to get rid of irrelevant terms in G. The next thing to notice is that all the spatial dimensions are equivalent here and therefore we only need to consider operators involving ∇. We now wish to eliminate the terms of the form $\nabla^{2n+1}h$ because of the symmetry of reflexion $x \to -x$. Under this transformation, the sign of all such terms would change, where the sigh of $\frac{\partial h}{\partial t}$ would stay the same, which is unacceptable. Finally, the last symmetry we will make use of is the fact that under the transformation $h \to -h$, the system behaves as if is was going back in time. This implies that $G(h) = G(-h)$. This gets rid of the terms of the form $(\nabla h)^{2n}$. At this point, we are left with

$$\frac{\partial h}{\partial t} = \eta(x, t) + F + \nabla^2 h + \nabla^4 h + (\nabla^2 h)(\nabla h)^2 + \ldots$$

To further simplify, we wish to get rid of the higher order terms. We are interested in the scaling exponents of the interface in the long term and large scale limits. To justify the neglect of higher order terms, we consider the transformation $x \to bx$ so that $h \to b^\alpha h$. Then we find that $\nabla^2 h \to b^{\alpha-2}\nabla^2 h$ and $\nabla^4 h \to b^{\alpha-4}\nabla^4 h$. More generally, every derivative corresponds to dividing by a b. In the limit where b is large, the higher order terms will be dominated by $\nabla^2 h$. The final form of the equation is called the Edwards-Wilkinson equation and is simply

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + F + \eta(x, t)$$

where ν and f are constants.

This stochastic equation allows us to solve for the different scaling exponents. We will not go in great details as to how to solve for them but only outline the main ideas and results. Following [NT92], the first step is to consider the Fourier transform in space and Laplace transform in time of the equation. This yields

$$h(k,\omega) = \frac{1}{\nu k^2 - i\omega}\eta(\mathbf{k},\omega)$$

where $k = \|k\|$ and

$$h(\mathbf{k},\omega) = \int e^{-i(\mathbf{k}\cdot x - \omega t)} h(x,t) d^d x dt$$

Because the noise $\eta(x,t)$ is not correlated, we get that the correlator in Fourier space is

$$< \eta(\mathbf{k},\omega)\eta(\mathbf{k}',\omega') >= 2D(2\pi)^d \delta^d(\mathbf{k}+\mathbf{k}') \int_0^\infty e^{i(\omega+\omega')t} dt$$

If we now use this to find variations in height in Fourier space we find

$$< h(\mathbf{k},t)h(\mathbf{k}',t) >= \frac{D(1-e^{-2\nu k^2 t})(2\pi)^d \delta^d(\mathbf{k}+\mathbf{k}')}{\nu k^2}$$

To find the width in real space, we need to take the inverse Fourier transform of this

$$w^2(L,t) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{d^d\mathbf{k}'}{(2\pi)^d} < h(\mathbf{k},t)h(\mathbf{k}',t) >$$

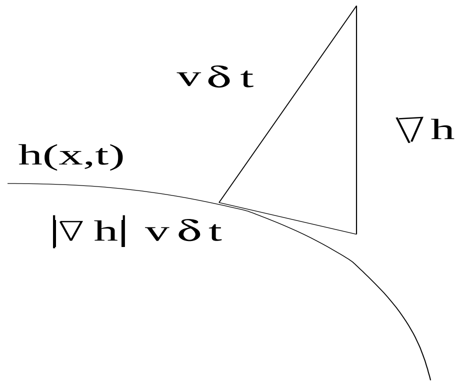
The limits of integration here are set by the size of the system L and by the size of a particle a such that $2\pi/L \leq k, k' \leq \pi/a$. Performing the angular integration gives

$$w^2(L,t) = K_d \int_{2\pi/L}^{\pi/a} \frac{dk}{k^{3-d}}\frac{D}{\nu}(1-e^{-2\nu k^2 t})$$

where K_d is a constant which depends on d only. Integrating by parts gives a rather complicated function of the form

$$< h(x,t)h(x',t') >= \frac{D}{2\nu}|x-x'|^{2-d} f\left(\frac{\nu|t-t'|}{|x-x'|^2}\right)$$

with the properties that $f(u) \to cst$ as $u \to \infty$ and $f(u) \approx u^{(2-d)/2}$ as $u \to 0$. This allows us to deduce that for $d = 1$, we find that $\alpha = 1/2$, $\beta = 1/4$ and $z = 2$. Notice also that in higher dimensions, both $\alpha = \beta = 0$ which means that diffusion dominates then and that the interface is basically flat.



4.2 Nonlinear systems

Let us now consider a slightly different system in which one of the numerous symmetries discussed previously is lost. The ballistic deposition (BD) model has the same set up as the random deposition, except that the particles now stick to the first neighbouring particle they encounter. Particles still come in vertically, but will now stop as soon as they reach the level of an adjacent column. This model is used to describe the formation of silicon chips where the attractive force between atoms is strong.

With this model, the up-down symmetry is lost since the shape of the interface depends very much on the direction particles come from. The rest of the argument leading to the Edwards-Wilkinson equation still holds, which leaves us with the same equation plus a nonlinear term

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda (\nabla h)^2 + F + \eta(x,t).$$

This equation is called the Kardar-Parisi-Zhang equation (KPZ).

There is actually another way to justify the presence of the nonlinear term. The growth of the interface in this system occurs roughly normal to the surface. If we suppose that the influx of particles is given by $v\delta t$, we get from (figure 4.2)

$$\frac{\delta h}{\delta t} = v(1 + (\nabla h)^2)^{1/2} \approx v + \frac{v}{2}(\nabla h)^2$$

It is possible, although very tedious to solve for the width of a surface obeying the KPZ equation for $d = 1$, and to find $\alpha = 1/2$, $\beta = 1/3$ and $z = 3/2$. However, for $d \geq 2$, there are no available analytic results. In the case of ballistic deposition, results from numerical simulations agree very well with the predicted exponents for d = 1.

5 Bacterial growth

We will present different models of bacterial growth. The physical situation under consideration is one where a colony of bacteria grows on a substrate from a single seed (or very few). The bacteria themselves are considered to be immobile, although the colony as whole will grow and change shape through reproduction.

Before we discuss the different existing models, it must be said that experiments exhibit an incredibly wide variety of shapes and patterns for bacterial growth. Most of theses differences come from the bacteria themselves, who have many characteristics that can influence their growth: their shape, the perception they have of other bacteria, their ability to figure out in which direction is the gradient of nutrients etc. The conditions in which the experiments are performed also have a large impact. The most spectacular patterns are usually obtained under stressed conditions, that is to say with restricted food or water, or in the presence of chemicals. It is therefore not surprising that so many models exist, and also that it is very often possible to find an experiments which seem to be described by a given model. However, a more scientific approach where the model is derived from biological features of the bacteria themselves and then compared to the experiment is much less successful, mostly because it is so difficult.

5.1 Eden Model

The simplest model, and the one that probably applies in most cases is the Eden model. Starting from an initial seed, a new particle is added to cluster on one of the surface sites. A surface site here is define as a site sharing a side with the existing cluster (no diagonals). The way in which the surface site is chosen can vary. One version of the Eden model selects with equal probability among all the surface sites where a new particle will be added (A). Another version counts the number of neighbours of each surface site and the probability that a new particle is added is directly proportional to the number of neighbours (B). The third version of the Eden model chooses a “mother cell” with equal probability among the particles which are not completely surrounded by other particles (C).

All three versions of the Eden model lead to similar looking “bacterial colonies”. The dimension of the resulting object is the same as the dimension of the space. Since all surface sites have more or less the same probability to be filled, it is very unlikely that a site near the seed will remain empty for a long time. In two dimensions, one obtains a shape which is roughly a disc, with a rough surface. It is difficult to characterize this surface in terms of the exponents mentioned earlier because the length of the object, taken to be the average radius is also changing with time. A way to get around this problem is usually to grow a cluster on a strip

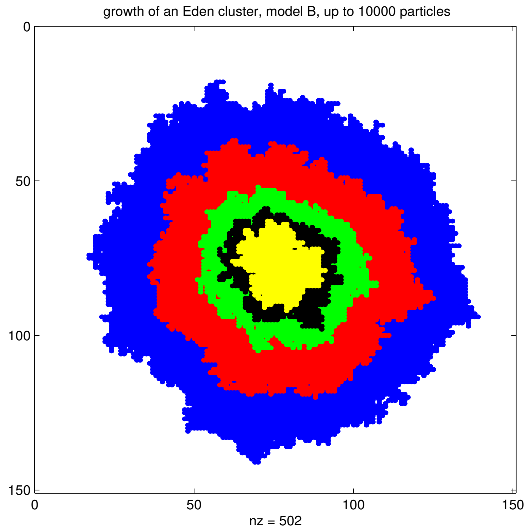


Figure 2: Growth of an Eden cluster of type B

of fixed length rather than from a seed. It is then a simple matter to compute the exponents α, β and z. These results can be thought to apply to the Eden model as long as the average height of the deposit is of the order of the length of the strip, which means that the radius is of the order of the circumference. In the strip geometry, it is found in two dimensions that $\alpha \approx 1/2$ and $\beta \approx 1/3$. This should hardly be surprising because, following our symmetry argument, the simplest continuous equation describing this phenomena is the KPZ equation, for which the exponents are exactly the same.

The figure (5.1) shows the evolution of an Eden cluster. This figure was obtained by a simulation on a 500 by 500 square grid. The object is clearly two dimensional. The exponents of the interface agree rather well with the ones mentioned above, not surprisingly.

The physical justification for the Eden model is rather simple. The bacteria are not supposed to be moving and so the only way the colony will grow is via offsprings. It is assumed that a bacteria will divide itself only if it has sufficient room to do so. This means that any bacteria which is completely surrounded is basically dead. Every variation of the Eden model has a biological justification, but the bottom line is that apart from the restrictions just mentioned, division can occur anywhere with equal probability. This model therefore does not take into account the conditions in which the bacteria grow. There is no shortage of food and no inhibition of growth by waste produced by other cells. This is a reasonable assumption in cases where food is abundant and the stress on the bacteria is low.

5.2 Diffusion-limited aggregation

The diffusion-limited aggregation model (DLA) is at the other end of the spectrum. This model has a very simple growth rule and has been studied in great details, [Vic92], [Mea98]. We start from a seed at the origin. A random walker is then released form a point taken uniformly on a circle of radius strictly greater than maximum radius of the aggregate. As soon as the walker comes in contact with the aggregate, it sticks. Of course, there is always the possibility that the walker gets lost and never actually hits the aggregate, in which case the process simply starts over with a new walker.

A natural question here is why would this model bacterial growth? To answer this question we first derive an equivalent way to construct a DLA cluster. Let us first fix a radius R such that the random walker is released from a circle of radius R. If we denote the number of walks that get to a point $\vec{r}$ after n steps by $Z_n(\vec{r})$, we have

$$Z_{n+1} = \sum_{k=1}^{C} Z_n(\vec{r} + k)$$

where C is the number of adjacent sites and k is a vector from an adjacent site. If we now call $u(\vec{r}, n)$ the probability for a walker to be at $\vec{r}$ after n steps, then we must have $u(\vec{r}, n) \propto Z_n(\vec{r})$. Combining these, we get

$$u(\vec{r}, n+1) - u(\vec{r}, n) = \sum_{k=1}^{C} u(\vec{r} + k, n) - u(\vec{r}, n)$$

What we are really interested in, however, is the probability that the walker is at a point $\vec{r}$ at some time. We thus define $P(\vec{r}) = \sum_n u(\vec{r}, n)$. Summing the previous equation gives us

$$u(\vec{r}, \infty) - u(\vec{r}, 0) = \sum_{k=1}^{C} P(\vec{r} + k) - P(\vec{r}).$$

The terms on the left hand side are much smaller than the ones on the right hand side and so will we neglect them. Notice also that the right hand side is a discretized version of a Laplacian. Therefore, the probability to find a walker at position $\vec{r}$ obeys Laplace's equation, with the boundary conditions that $P = 0$ on all the occupied sites, and $P = \kappa$ on a circle of radius R, where κ is a non-zero constant. The probability to grow a new particle at a point $\vec{r}$ is given by the probability to find a walker there, which because of our boundary condition is proportional to the derivative of P in a direction normal to the surface.

We thus have an equivalent way to grow a DLA cluster. At each time step, we need to solve Laplace's equation with the boundary conditions mentioned above, and a new particle is added with probability proportional to the normal derivative. Is has been verified in simulations that these processes give rise to similar structures.

Knowing this, it is now possible to find physical applications for this model. The most important one is probably the dielectric breakdown, [aLP84]. In this problem, a region of space is initially charged, and it expands by charging neighbouring particles. The application is obvious since the electric field has to obey Laplace's equation. The boundary condition in the charged region is simply that the field is constant. The boundary condition at infinity can vary though, depending on the circumstances. But in any case, DLA-like structures are observed in experiments studying dielectric breakdowns.

One can also apply the DLA model to bacteria, in a certain regime. Let us first assume that the food the bacteria need is undergoing some regular diffusion process. It therefore obeys the heat equation. The boundary conditions are the same as in the DLA model since the food concentration is zero where there are bacterias, and is constant if we go far enough from the growing cluster. In order to get the DLA model, we must now suppose that diffusion is much faster than cell division so that the food concentration reaches its equilibrium configuration between each cell division. The concentration of food therefore obeys Laplace's equation as well and we recover the DLA model.

There is however a wide range of experiments which lie somewhere in between those two extremes. We propose here a model that attempts to bridge between them. This model is similar to the Eden model, but with two parameters which can be tuned according to the experiment. Ideally, the experiment itself would provide the values of those parameters, from the anatomy of the bacteria and from the conditions. However, for lack of biological information, we will simply study the effect that these parameters have on the resulting colony.

The algorithm to grow our colony is the following. For each unoccupied site on the perimeter of the cluster, we compute two quantities. The first one is the number of immediate neighbours N. The higher the number of neighbours, the better the chance that this site will become occupied since it has many potential "mother cells". The second quantity is the density of occupied sites in a certain radius, δ_R. The density is simply taken to be the number of occupied sites within a circle of radius R. The higher the density, the smaller the chances to grow a new particle since there will be less food left in a crowded area. We then grow a new site at $\vec{r}$ with probability p, where $p \propto N/\delta_R^\beta$. The two parameters we can play with are the radius and the exponent β. The radius is a measure of how fast bacteria reproduce compared to diffusion of food, and β simply attributes more or less importance to the density of the surroundings. In a situation where there is abundance of food, the density will not matter very much since there will be enough food anyway. However, as food rarefies, β should increase, suggesting that only the bacteria in the most favorable positions will reproduce.

Certain limits of this model yield structures which are easy to predict. The case $R = 0$ simply corresponds to the Eden model of type (B), where only the number of neighbours matter. In the case $R = 1$ and $\beta = 1$, the density term cancels exactly the neighbour term and we get the Eden model of type (A), where all the surface sites are equally likely to grow. Also, is we take the limit $R \to \infty$, then all points experience the same density and this is equivalent to the case $R = 0$. Finally, taking the limit $\beta \to \infty$, we find that the site with the smallest density will always be chosen. If the radius is greater than one, this will give a straight line. For $R = 1$, this gives a compact structure with holes of size 1. Effectively, any point can only have one occupied neighbour.

6 Results and conclusion

One therefore expects a transition between the two regimes. More precisely, there should be a transition between a two dimensional object given by the Eden model, and a one dimensional object found for large β and $r \geq 2$. This is indeed seen in our results. The following figures show the changes in the shape of the cluster as β increases, for fixed values of R. The cluster progressively becomes more and more linear. For the case $R = 2$, the critical value of β appears to be around 8. It is difficult to make a precise statement about the critical exponent since the dimension is difficult to measure. Moreover the objects do not appear to be self-similar, at least not in the scales considered here. There is clearly a length scale set by the radius which prevents self-similarity. We have restricted ourselves to small values of R to reduce the amount of computation involved, and because the trend is seen rather easily with small R. For small values of β, the results are very much like Eden clusters, with the distinction that holes of the size of R appear for $R \geq 2$. The resulting object is thus still two-dimensional, but it only occupies a fraction of space.

More interesting is the transition to one-dimensional objects as β increases. All the figures shown here are simulations made with 5000 particles on a square grid. The scale of the plot changes to allow a better visualization of the result. In some cases, boundary effects are present. The outside of the box is taken here to be empty from particles, but inaccessible. This explains why certain simulation seem to bounce off the side of the box. A more complete characterization of the dimension of the different results is possible, as well as a more detailed analysis of

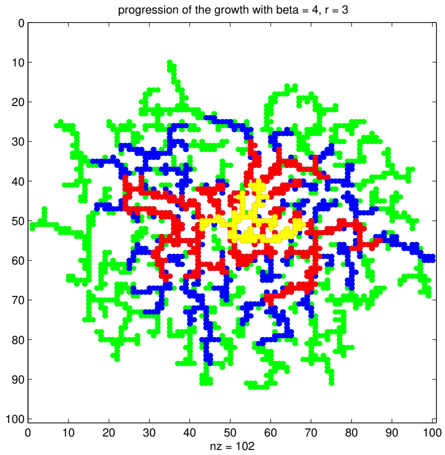
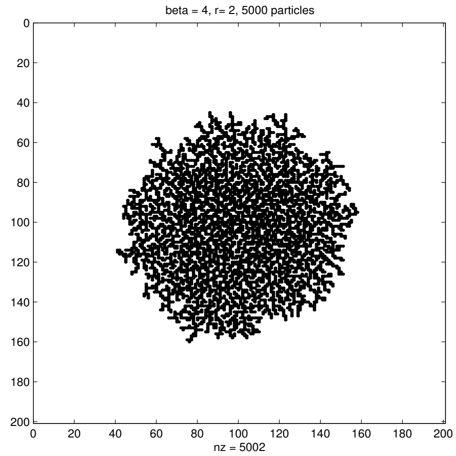
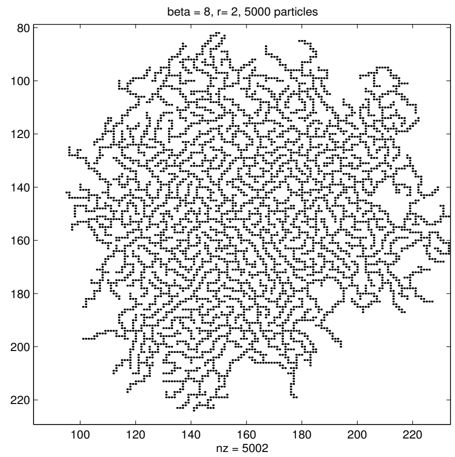


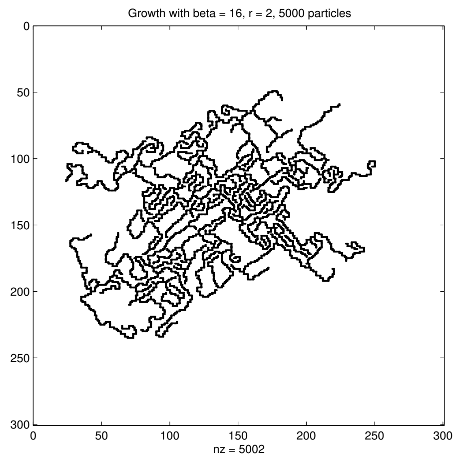
Figure 3: evolution of a cluster with $\beta = 4$, $r = 3$

the critical exponents. However, the purpose of this report was simply to look for a smooth transition between the Eden model and a DLA model. In this respect, this is only a half success since the DLA model is never reached with this model. However, there is indeed a transition between two-dimensional objects and one-dimensional objects. Notice that instead of looking like DLA cluster, the results of simulations with large β resemble self-avoiding walks. This is reasonable since our model is in fact modeling a "self-disliking" walk. It sometimes crosses itself, but with very low probability.

Another approach to the problem would be to consider explicitly the diffusion of food. Of would then have to solve the heat equation with the boundary conditions mentioned for the DLA. The diffusion constant would then be the only parameter of the model, if the rate of cell division is fixed. In the limit $D \to 0$, one would expect to recover the Eden model, while in the limit $D \to \infty$, one should recover the DLA model. Unfortunately, this seems like a much harder simulation to perform and it has not been attempted here. It would be extremely interesting to compare the results of such simulations with the results presented here.


beta = 4, r= 2, 5000 particles
nz = 5002


beta = 8, r= 2, 5000 particles
nz = 5002


Growth with beta = 16, r = 2, 5000 particles
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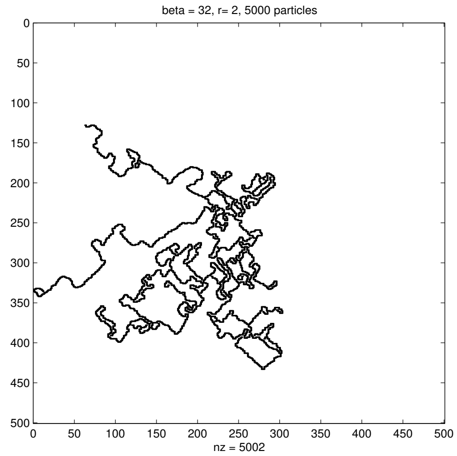
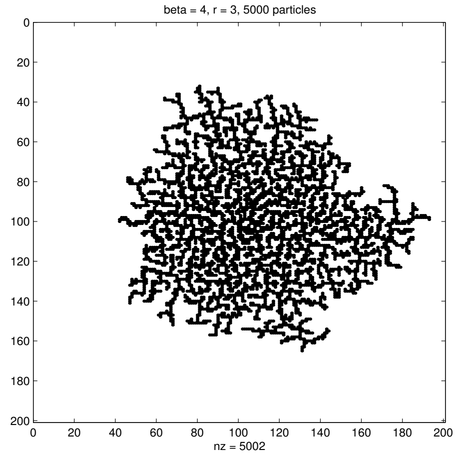
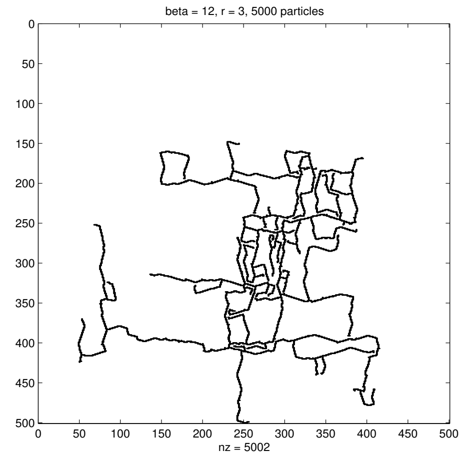
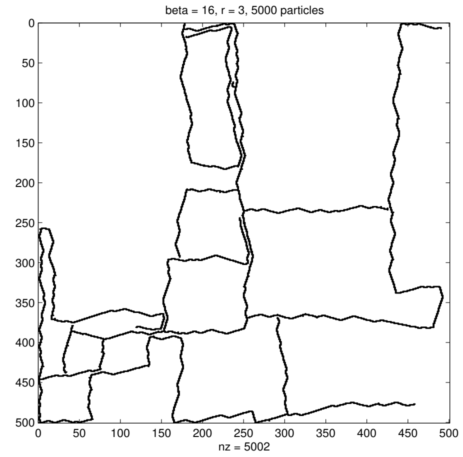

beta = 32, r= 2, 5000 particles
nz = 5002


beta = 4, r = 3, 5000 particles
nz = 5002


beta = 8, r = 3, 5000 particles
nz = 5002



beta = 12, r = 3, 5000 particles
0
50
100
150
200
250
300
350
400
450
500
0
50
100
150
200
250
300
350
400
450
500
nz = 5002



beta = 16, r = 3, 5000 particles
0
50
100
150
200
250
300
350
400
450
500
0
50
100
150
200
250
300
350
400
450
500
nz = 5002

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