18.325 Topics in Applied Mathematics, Spring 2001, Prof. Bazant.

Problem Set 3

Due at lecture on Thursday, May 10.

1. **Percentile Order Statistics and the Tail of the Bernoulli Random Walk.**

   Let X_i $(i = 1, ..., N)$ be independent, identically distributed, continuous random variables with cumulative distribution function (c.d.f.) $P(x) = \text{Prob}(X_i \leq x)$ and probability density function (p.d.f.) $p(x) = dP/dx$. Let $Y_N^{(\alpha)}$ be the 100αth percentile of the N samples (e.g. $\alpha = 0.95$ for the 95th percentile), which is uniquely defined by ordering the outcomes: $X_{(1)} < X_{(2)} < \ldots < X_{(N)}$, using the standard notation of *order statistics*. We then choose $Y_N^{(\alpha)} = X_{([\alpha N])}$, where $[x]$ denotes the nearest integer to x.

   (a) Derive an expression for the PDF $f_N(y)$ of $Y_N^{(\alpha)}$ which is related to the PDF of a Bernoulli random walk with probability $P(y)$ of "stepping to the right". Explain why the PDF is sampled far outside the "central region".

   (b) Using the results of the saddle-point analysis from lecture 16, derive a globally-valid asymptotic expression for $f_N(y)$ as $N \to \infty$. (First replace $[\alpha N]$ by αN, which can be rigorously justified in this limit.)

   (c) Show that a sort of "CLT" holds in the "central region": The variable $(Y_n - y_\alpha)/\sigma_{Y_N}$ has an asymptotic normal distribution, where the mean is y_α, the parent percentile defined by $P(y_\alpha) = \alpha$, and the asymptotic variance is

   $$\sigma_{Y_N}^2 = \frac{\alpha(1-\alpha)}{Np(y_\alpha)^2}.$$

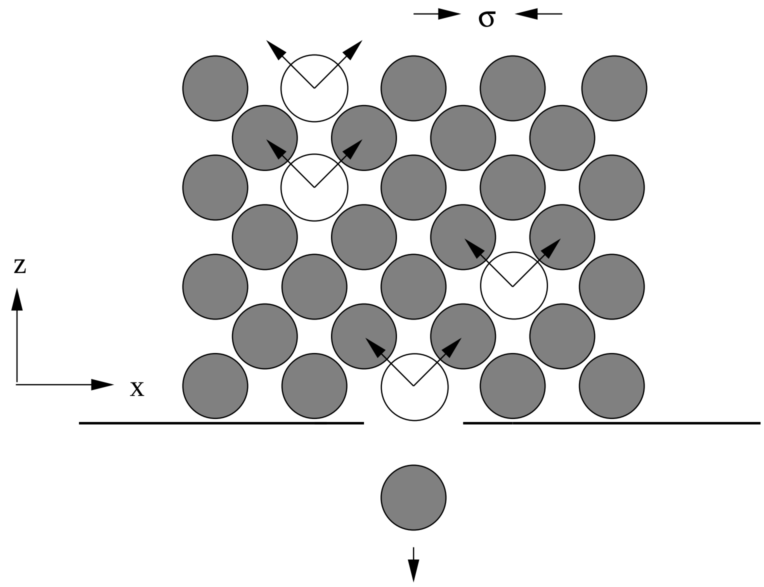
   (d) Suppose that X_i are iid standard normal random variables (mean 0, variance 1). Compare the limiting distributions of the mean $Z_N = \sum_{n=1}^{N} X_n/N$ and the median $Y_N^{(0.5)}$. Which has a bigger variance?

   Order statistics have many applications. For example, the MIT admissions committee may wonder whether or not GRE Mathematics test scores from different years can be meaningfully compared. This could be done by seeing whether various percentiles of actual scores are consistent with the hypothesis of stationary, random fluctuations from year to year. Order statistics are also used to price Internet bandwidth, which is typically billed in proportion to the 95th percentile of data-rate samples taken at regular intervals over a one month billing period. (The present market cost is roughly \$500/(Mbit/sec).)

2. **Saddle-Point Asymptotics for Another "Thin-Tailed" Random Walk.**

   Consider a random walk with iid displacements with PDF $p(x) = a\text{sech}(bx)$.

   (a) Find the structure function $\hat{p}(k)$, and choose a and b so that the steps are properly normalized and have unit variance.



(b) Use the saddle point method to determine the leading-order asymptotics of the PDF $P_n(x)$ of the position of the walker after n steps, in the limit $n \to \infty$ with $\xi = x/n$ fixed.

(c) What is the leading-order behavior of $P_n(n\xi)$ very far into the tail, as $\xi \to \infty$?

(d) Take the limit $\xi \to 0$, with $\zeta = \sqrt{n}\xi = x/\sqrt{n}$ fixed, in the expression from part (b), and show that the Central Limit Theorem holds.

3. **Slowly Draining Sand.**

Consider close-packed sand draining from a silo through a narrow hole at the bottom, described in class. Simplify to the "2d" case of a narrow, flat silo. Model the mass transport by a directed random walk of voids upwards from a point source at the bottom on a triangular lattice, as shown in the figure. At every time step of duration τ, each void moves up by one layer in the lattice with equal probability of going left or right by a distance σ, as shown. At every kth time step ($k \geq 1$), a new void is introduced at the bottom hole (as a grain of sand falls out of the container).

(a) **An Infinite Container.** Take the continuum limit, and solve for the steady-state concentration of voids $c(x, z)$, where $x = 0$ is the location of the hole at the bottom. (The mass density is $c_{max} - c$.) Also solve for the steady-state mass velocity profile $\vec{v}(x, z)$ of the draining sand. What is the shape of the "central region" where the sand is moving significantly? Compare (qualitatively) the "flow" of sand in this model with with the case of a draining fluid in the same geometry.

(b) **A Parabolic Container.** Now suppose that the container has a parabolic shape $z_c = ax^2$. Determine the resulting steady-state void concentration and velocity field.

4. **Sorption and Desorption from a Lattice.**

Interacting random walkers on a lattice are a reasonable model for many physical situations, like the discharging and charging of a rechargeable lithium battery electrode. "Desorption" is the case of a half simple-cubic lattice initially full of random walkers which is being drained (constant concentration of zero at the surface), and "sorption" is the case where an initially empty lattice is filled (by a constant concentration $= c_{max}$ at the surface). A crucial point is that walkers have an excluded volume interaction (i.e. they cannot share the same site.)

(a) **Nonlinear Diffusion Approximation.** This situation should be amenable to a continuum approximation of the form:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D(c)\frac{\partial}{\partial x}\right)$$

where the bare diffusion constant D_o has been replaced by a concentration-dependent diffusion coefficient $D(c) = D_o(1 - c/c_o)$, where c_o is the maximum concentration possible in the lattice. (Why?) Show that similarity solutions ($x > 0, t > 0$) are possible for the cases of desorption ($c(x,0) = 1$, $c(0,t) = 0$) and sorption ($c(x,0) = 0$, $c(0,t) = 1$), and sketch what you think the solutions might look like. There may be "fronts" where $\partial c/\partial x$ is discontinuous...

(b) **Simulations.** (EXTRA CREDIT) Simulate sorption and desorption from a two-dimensional square lattice. Consider "blind" walkers, who at each time step (e.g. in a random order) pick a random direction, and simply do not move if the neighboring site in that direction is already occupied. Do your simulations agree with the predictions of the mean-field, nonlinear diffusion model? (You may need to solve the similarity equations numerically.)