Solutions to Problem Set 2a

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March 20, 2001

1 Inelastic diffusion

(a) X_N is the sum of independent random variables, scaled from values drawn from identical distributions. The steps are $\Delta_n = a^n \Delta X_n$, therefore the transition probability of Δ_n is

$$p_n(x) = \frac{1}{a^n} p_0\left(\frac{x}{a^n}\right) \tag{1}$$

as a function of $p_0(x)$, the p.d.f of the iid steps. Its structure function is

$$\hat{p}_n(k) = \hat{p}_0(a^n k) \tag{2}$$

by scaling properties of the Fourier transform. Using the known cumulants of $p_0(x)$,

$$\log \hat{p}_n(k) = \log \hat{p}_0(a^n k) = -c_2 \frac{(a^n k)^2}{2} - ic_3 \frac{(a^n k)^3}{3!} - c_4 \frac{(a^n k)^4}{4!} - \cdots \tag{3}$$

from which the cumulants of $p_n(k)$ can be seen to be $c_{n,l} = a^{nl} c_l$.

(b) Since the steps are independent, the cumulants of $\hat{P}_N(k)$ can be obtained from

$$\log \hat{P}_N(k) = \sum_{m=1}^{N} \log \hat{p}_m(k) \tag{4}$$

then

$$C_{N,l} = \sum_{m=1}^{N} c_{m,l} = \sum_{m=1}^{N} a^{ml} c_l \tag{5}$$

(c) In the limit $N \to \infty$, for $0 \leq a < 1$, the above series converges to the sum of a geometric series

$$C_l = \lim_{N \to \infty} C_{N,l} = \frac{a^l}{1 - a^l} c_l \tag{6}$$

which can be written, for $a = 1 - \epsilon$, as

$$C_l = \frac{(1-\epsilon)^l}{1-(1-\epsilon)^l} c_l \tag{7}$$

From this results, the normalized cumulants tend to

$$\lim_{\epsilon\to 0} \frac{C_{2m}}{C_2^m} = \lim_{\epsilon\to 0} \frac{(1-\epsilon)^{2m}}{2m\epsilon} \left(\frac{2\epsilon}{(1-\epsilon)^2}\right)^m \frac{c_{2m}}{c_2^m} = \frac{2^{m-2}}{m}\epsilon^{m-1}\frac{c_{2m}}{c_2^m} = \epsilon^{m-1}\Lambda_{2m} \tag{8}$$

as $\epsilon \to 0$. In this limit, $\Lambda_{2m} = \epsilon^{1-m}\frac{C_{2m}}{C_2^m} = 2^{m-2}c_{2m}/mc_2^m$.

(d) If $\phi(\zeta, \epsilon) = C_2^{1/2} P_\infty(\zeta C_2^{1/2})$, defining $\omega = kC_2^{1/2}$, then the transform of $\phi(\zeta,\epsilon)$ is

$$\log \hat{\phi}(\omega) = -\frac{\omega^2}{2} - i\epsilon^{1/2}\Lambda_3\frac{\omega^3}{3!} - \epsilon\Lambda_4\frac{\omega^4}{4!} - \cdots \tag{9}$$

where $\Lambda_{2m} = \epsilon^{1-m}\frac{C_{2m}}{C_2^m}$. Taking the inverse transform,

$$\phi(\zeta) = \int_{-\infty}^{\infty} e^{i\omega\zeta} e^{-\frac{\omega^2}{2} - i\epsilon^{1/2}\Lambda_3\frac{\omega^3}{3!} - \epsilon\Lambda_4\frac{\omega^4}{4!} - \cdots} \frac{dk}{2\pi} \tag{10}$$

where the terms of order $l > 3$ have coefficients of order $O(\epsilon^{m-1})$. Since the case $\epsilon \to 0$ is being studied, the exponential can be expanded around $-\omega^2/2$ as

$$\phi(\zeta) = \int_{-\infty}^{\infty} e^{i\omega\zeta} e^{-\frac{\omega^2}{2}} \left(1 - i\epsilon^{1/2}\Lambda_3\frac{\omega^3}{3!} - \cdots\right)\frac{dk}{2\pi} \tag{11}$$

From this expression,

$$\phi(\zeta) = \phi_0(\zeta)\left(1 - \epsilon^{1/2}\frac{\Lambda_3}{3!}\frac{\partial^3\phi_0(\zeta)}{\partial\zeta^3} + \cdots\right) = \phi_0(\zeta)\left(1 + O(\epsilon^{1/2})\right) \tag{12}$$

which shows that the distribution tends to a gaussian $\phi_0(\zeta)$ as $\epsilon \to 0$.

(e) Continuing the process outlined in the previous item and evaluating the derivatives of the gaussian, it is found the ordinary Gram-Charlier expansion

$$\begin{aligned}\phi(\zeta) = \phi_0(\zeta)\Bigg[1 - \epsilon^{1/2}\frac{\Lambda_3}{3!}(3\zeta - \zeta^3) + \\ \epsilon\left(\left(\frac{\Lambda_4}{8} - \frac{5\Lambda_3^2}{24}\right) - \left(\frac{\Lambda_4}{4} - \frac{5\Lambda_3^2}{8}\right)\zeta^2 + \left(\frac{\Lambda_4}{24} - \frac{5\Lambda_3^2}{24}\right)\zeta^4 + \frac{\Lambda_3^2}{72}\zeta^6\right)\Bigg]\end{aligned} \tag{13}$$

(f) The approximation of the previous item is compared with numerical simulations. The simulations consisted in simple random walks, i.e. $p_n(\Delta_n) = \frac{1}{2}[\delta(x-a^n) + \delta(x+a^n)]$, for $a = 1-\epsilon$, and were obtained using $2\cdot 10^6$ walks with 50 bins spanning the maximum expected distance of a walker. The results are shown in Figures 1 to 3. It is noticed that the approximation is very good, even for large ϵ. The correction is also plotted for reference. Notice that for this walk, $\Lambda_3 = 0$ by symmetry, therefore

$$\phi(\zeta) = \phi_0(\zeta)\left(1 + \frac{\epsilon\Lambda_4}{8} - \frac{\epsilon\Lambda_4}{4}\zeta^2 + \frac{\epsilon\Lambda_4}{24}\zeta^4\right) \tag{14}$$

where $\zeta = x/C_2 = x\frac{1-a^2}{a^2}$.

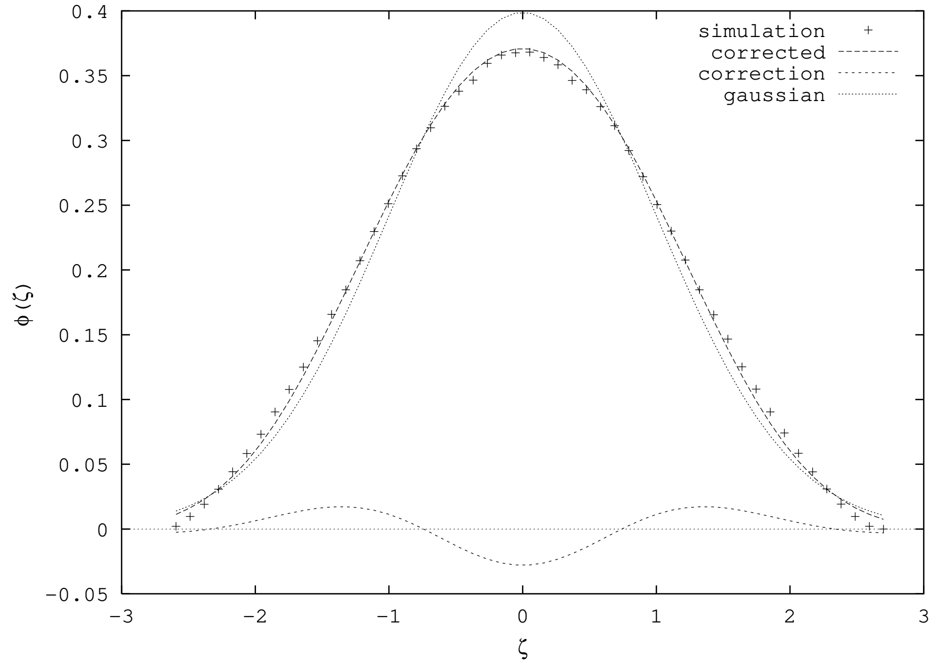

Figure 1: Approximation to the inelastic diffusion problem ($\epsilon = 0.25$).

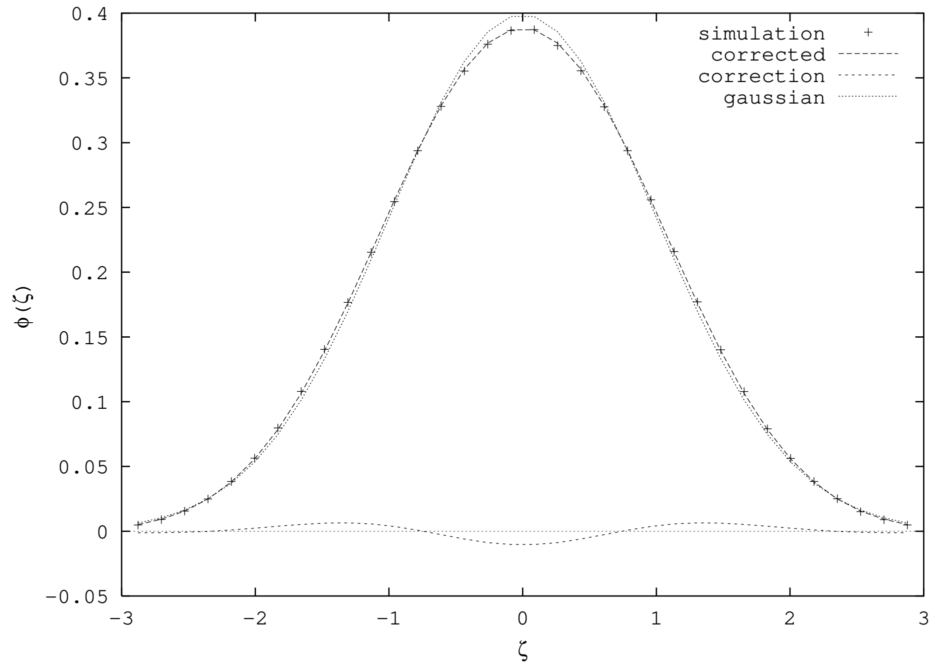

Figure 2: Approximation to the inelastic diffusion problem ($\epsilon = 0.1$).

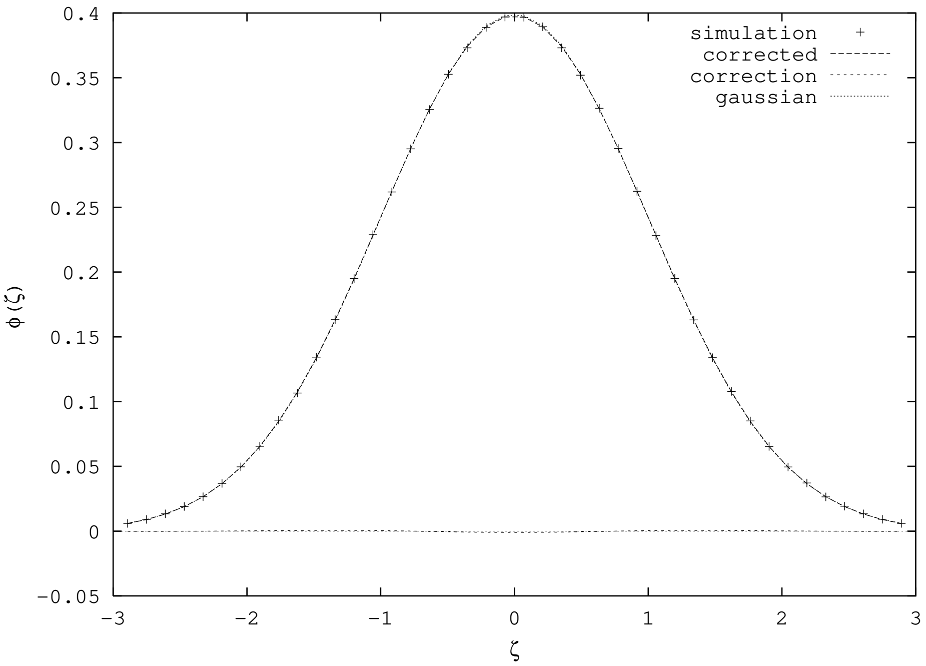

Figure 3: Approximation to the inelastic diffusion problem ($\epsilon = 0.01$).

2 Normal diffusion with Fat-tailed transitions

The structure function for $p(x) = A/(1+x^2)^2$ is

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} \frac{A}{(1+x^2)^2} dx = \frac{A\pi}{2} e^{-|k|}(1+|k|) \tag{15}$$

and, since $\hat{p}(0) = 1$, $A = 2/\pi$. From this function, the probability distribution after n steps is

$$\hat{P}_n(k) = e^{-n|k|}(1+|k|)^n \tag{16}$$

ans its logarithm is

$$\log \hat{P}_n(k) = -n|k| + n\log(1+|k|) \tag{17}$$

$$= -n|k| + n|k| - \frac{n}{2}k^2 + \frac{n}{3}|k|^3 - \frac{n}{4}k^4 + \cdots \tag{18}$$

$$= -\frac{n}{2}k^2 + \frac{n}{3}|k|^3 - \cdots \tag{19}$$

this is a special case of the expression for fat-tailed distributions, solved in class for arbitrary coefficients of $|k|^3$. Since the first term is $-nk^2/2$, a gaussian is obtained, and its coefficient is the standard deviation $\sigma = 1$. The next term indicates that the third moment diverges, and its presence introduces functions of the Dawson integral in the Gram-Charlier expansion, but as $n \to \infty$ with fixed ζ, a gaussian is still obtained.

3 Anomalous diffusion with very fat-tailed transitions

From $p(x) = A/(1+x^2)$, the structure function is

$$\hat{p}(k) = A\pi e^{-|k|} \tag{20}$$

and from $\hat{p}(0) = 1$, $A = 1/\pi$. The transform of the distribution after n steps is

$$\hat{P}_n(k) = e^{-n|k|} \tag{21}$$

and inverting the transform, using the scaling property $a^{-1}f(x/a^{-1}) = \hat{f}(ak)$,

$$P_n(x) = \frac{1/n\pi}{1 + x^2/n^2} = \frac{n}{\pi(n^2 + x^2)} \tag{22}$$

which is clearly not a gaussian even if $n \to \infty$, as expected from the fact that the distribution has not a finit second moment. A series representation of this function is

$$P_n(x) = \frac{1}{\pi}\left(\frac{1}{n} - \frac{x^2}{n^3} + \frac{x^3}{n^4} - \frac{x^4}{n^5} + \cdots\right) \tag{23}$$

and a convenient scaling is $\zeta = x/n$, since

$$nP_n(\zeta n) = \phi(\zeta) = \frac{1}{\pi}\left(1 - \zeta^2 + \zeta^4 - \zeta^6 + \cdots\right) \tag{24}$$

therefore the distribution expands with N^α, where $\alpha = 1 > 1/2$, this being a super-diffusion problem.