6.S979 Lecture 8

Pset I due 10/2

Errata:

$2d \min_\theta \| e^{i\theta}|\psi\rangle - |EPR\rangle \|$

3. GHZ game

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$iXZ = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$Y^2 = I$, $YX = -XY$
$YZ = -ZY$

$(X \otimes X)(Z \otimes Z) = -Y \otimes Y$

– Last time $\sum_{i \in \mathbb{Z}_d} |-i\rangle\langle i|$

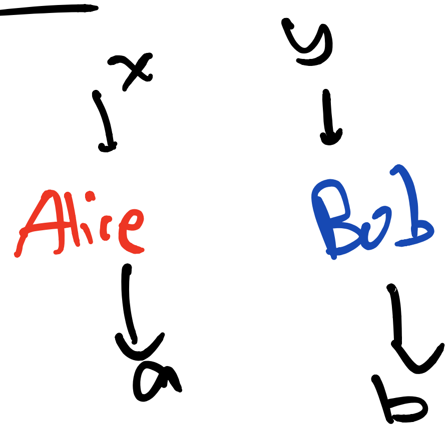
~~$|i + 2^{-1}d\rangle\langle i|$~~

$|d_{-}^{\psi} i\rangle\langle i|$

Quantum correlation sets

Correlation is

$p(a,b|x,y)$



Classical correlations:

$$p(a,b|x,y) = \sum_{\lambda\in\Lambda} \mu(\lambda)\, p_A(a|\lambda,x) \cdot p_B(b|\lambda,y)$$

$$C_{\text{classical}} = \{ p(a,b|x,y) \text{ classical} \} \in \mathbb{R}_+^{|X|\times|Y|\times|A|\times|B|}$$

convex set

Quantum correlations

$$p(a,b|x,y) = \langle \psi | A_a^x \otimes B_b^y | \psi \rangle$$

$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\mathcal{H}_A \cong \mathbb{C}^{d_A}, \quad \mathcal{H}_B = \mathbb{C}^{d_B}$$

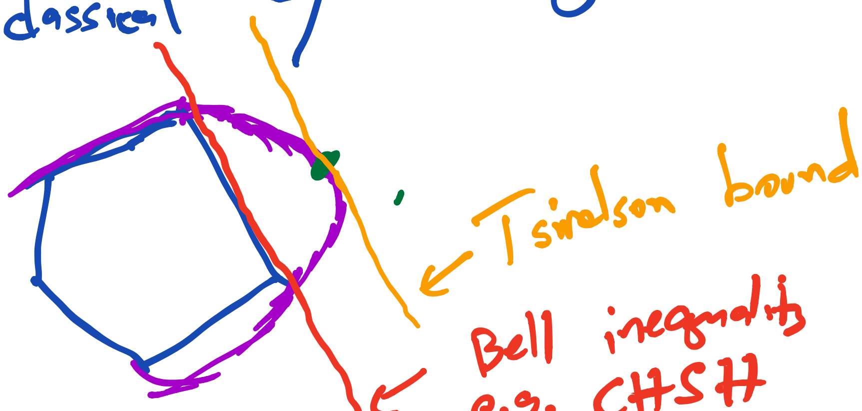
$$\forall x, \ \sum_a A_a^x = I, \quad (A_a^x)^2 = A_a^x$$

...

$$C_q = \{ p(a,b|x,y) = \langle \psi | A \otimes B | \psi \rangle \}$$

also convex

$$C_{classical} \subsetneq C_q$$



(in $\mathbb{R}^8$ for CHSH)

$$\frac{1}{4}\sum_{x,y,a,b} \mathbb{1}[a \oplus b = xy]\, p(a,b|x,y) \le 3/4$$

$\forall\, p \in C_{\text{classical}}$

$$\frac{1}{4}\sum_{x,y,a,b} \mathbb{1}[a \oplus b = xy]\, p(a,b|x,y) \le \cos^2(\pi/8)$$

$\forall\, p \in C_{qs}$

Classical set is a polytope
vertices are deterministic strategies

In general, $|X|=|Y|=|A|=|B| = n$

Then $C_{\text{classical}}$ has $\exp(n)$ vertices

bounded in n

What about C_q?

- No bound on $\dim(\mathcal{H}_A \otimes \mathcal{H}_B)$ as a fun. of n

- Even for constant n, C_q could have unboundedly many faces

- May not even be a <u>closed</u> set!

There are multiple possible quantum sets

1) C_q : finite dimensional correlations

2) C_{qs} : "spatial correlations"

$C_{qs} = \{ p(a,b|x,y) = \langle \psi | A_a^x \otimes B_b^y | \psi \rangle \}$

$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

can be ∞-dimensional

"Hilbert space"

Ex: Space of all square integrable functions on $\mathbb{R}$ $f: \mathbb{R} \to \mathbb{C}$

s.t. $\int |f(x)|^2 dx \leq \infty$

$\langle f, g \rangle = \int \overline{f(x)} g(x) \, dx$

Physical example: e.g. particle on a line

C_{qs} not necessarily closed either!

3) C_{qa} "approximable"

$$C_{qa} = \overline{C_q} = \overline{C_{qs}}$$

$p \in C_{qa}$ if $\forall \varepsilon$,
$\exists \tilde{p} \in C_q$ s.t. $|p - \hat{p}| \leq \varepsilon$

In practice can only learn p up to finite precision
so C_{qa} "more physical"

4) C_{qc} "quantum commuting"

~~$|\psi\rangle = \mathcal{H}_A \otimes \mathcal{H}_B$~~

$|\psi\rangle \in \mathcal{H}_{AB}$

"Alice operations" must commut w/ "Bob operations"

$$C_{qc} = \{p(a,b|x,y) = \langle\psi| A^x_a \cdot B^y_b |\psi\rangle\}$$

$|\psi\rangle \in \mathcal{H}_{AB}$, $A^x_a \overset{?}{=} A^{x}_a$, $\sum_a A^x_a = I$

...

$\forall x,y,a,b \quad [A^x_a, B^y_b] = 0$

<u>Fact:</u> If $\dim(\mathcal{H}_{AB}) < \infty$,

then $\mathcal{H}_{AB} = \bigoplus_i (\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i})$

$$A^x_a = \begin{pmatrix} A^x_{a,1} \otimes I & \\ & A^x_{a,2} \otimes I \end{pmatrix}$$

$$B^y_b = \begin{pmatrix} I \otimes B^y_{b,1} & \\ & I \otimes B^y_{b,2} \end{pmatrix}$$

$\Rightarrow$ $C_{qc,\, \dim < \infty} = C_{q}$

$$C_{classical} \subsetneq C_q \subseteq C_{qs} \subseteq C_{qa} \subseteq C_{qc}$$

$$C_{qa} = \overline{C_{qs}} = \overline{C_{qa}}, \quad C_{qc} = \overline{C_{qc}}$$

1) Which inclusions are strict?

2) Is there a nice description of any of these?

History:

1) Bell '64 $C_{classical} \neq C_q$

2) Tsirelson '80s found a description of a restricted class of correlations (XOR corr.)

- Hyperplane tangent to C_q

- Raised the question

$$C_{qc} \stackrel{?}{=} C_q \stackrel{?}{=} C_{qs} \stackrel{?}{=} C_{qa}$$

3) Scholz & Werner '08 + others realized that this is open!

$(C_{qa} = \overline{C_{qs}} = \overline{C_q})$

Junge & others

$C_{qc} \overset{?}{=} C_{qa}$ ⟵ Connes embedding problem

↑

"Tsirelson's problem"

4) Solved

Ji N. Vidick Wright Yuen

$C_q \neq C_{qs} \neq C_{qa} \neq C_{qc}$

↑ Coladangelo Stark

↑ Slofstra (also $C_q \neq C_{qa}$)

No computationally tractable

description of these sets

One takeaway:

Correlations can reveal a lot about the physical model

e.g. finite vs. infinite dim

Q: Can we reverse-engineer QM from C_{qa}

Tsirelson's characterization of XOR correlations

$p(a, b \mid x, y)$

$a, b \in \{\pm 1\}$

↘ "XOR correlation matrix"

$$C_{xy} = \sum_{a,b} (a \cdot b)\, p(a,b|x,y)$$

$$= \langle \psi | A^x B^y | \psi \rangle$$

$$A^x = (A_0^x - A_1^x) \qquad B^y = (B_0^y - B_1^y)$$

$$\forall\, x,y \quad -1 \le C_{xy} \le 1$$

$x \in X$, $y \in Y$ w.l.o.s. $X \cap Y = \emptyset$

$$\begin{array}{c c} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} C_{xx'} & C_{xy} \\ C_{yx} & C_{yy'} \end{pmatrix} \end{array}$$

o) $C_{yx} = C_{xy} = \langle \psi | A_x B_y | \psi \rangle$

$$C_{xx'} = \langle \psi | A_x A_{x'} | \psi \rangle$$

$$C_{yy'} = \langle \psi | B_y B_{y'} | \psi \rangle$$

$$\notin \mathbb{R}$$

1) $C_{xx} = C_{yy} = I$ $\quad \forall x, y$

$A_x^2 = I \quad B_y^2 = I$

2) $C_{xx'} = \overline{C_{x'x}}$

$\langle \psi | A_x A_{x'} | \psi \rangle = \overline{\langle \psi | A_{x'} A_x | \psi \rangle}$

$$C^\dagger = C$$

3) $-1 \leq C_{xy} \leq I$

4) $C \succeq 0$

Pf: Remember pf #1 of Tsirelson bound

$u_x = A_x|\psi\rangle, \; u_y = B_y|\psi\rangle$

$C_{ij} = \langle u_i, u_j \rangle$

"Gram matrix" $\Rightarrow$ PSD

$$x^\dagger C x = \sum_{ij} \bar{x}_i C_{ij} x_j$$

$$= \sum_{ij} \bar{x}_i \langle u_i, u_j \rangle x_j$$

$$= \left\langle \left(\sum_i x_i u_i\right), \left(\sum_i x_i u_i\right) \right\rangle$$

$$= \left\| \sum_i x_i u_i \right\|^2 \geq 0$$

Exercise:

Show $C \succeq 0 \Rightarrow |C_{xy}| \leq 1$

$C_{xx} = C_{yy} = 1$