6.S979 Lecture 4

Last time:

3 proofs of Tsirelson bound

$\omega^*_{CHSH} \leq \cos^2(\pi/8)$

Saturating bound

$\Rightarrow$ CHSH: $A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$

$(\cdot p)^2 + (q)^2 = 4I - \sqrt{2}\,CHSH$

$\langle\psi| p^2 |\psi\rangle = \langle\psi| q^2 |\psi\rangle = 0$

$\Rightarrow p|\psi\rangle = q|\psi\rangle = 0$

$$\left(A_{0/1} \otimes I - I \otimes \frac{B_0 \pm B_1}{\sqrt{2}}\right)|\psi\rangle = 0$$

$\Rightarrow A_0 A_1 \otimes I|\psi\rangle = -A_1 A_0 \otimes I|\psi\rangle$!

Also showed that

$A_0 A_1 = -A_1 A_0 \Rightarrow \exists\, \mathcal{U}$ s.t.

$\mathcal{U}^\dagger A_0 \mathcal{U} = Z \otimes I$

$\mathcal{U}^\dagger A_1 \mathcal{U} = X \otimes I$

Interlude: Schmidt decomposition

$$|\psi\rangle = \sum_{i,j=1}^{d} \psi_{ij}\, |i\rangle \otimes |j\rangle$$

$$= \sum_{k=1}^{r} \sigma_k\, |u_k\rangle \otimes |v_k\rangle$$

$r \leq d$

$\sigma_k \geq 0$

$\{|u_1\rangle \ldots |u_r\rangle\}$ is orthonormal

$\{|v_1\rangle, \ldots |v_r\rangle\}$ orthonormal

Pf: Recall SVD

$$|\psi\rangle = \sum_{ij} \psi_{ij} |i\rangle \otimes \underbrace{|j\rangle}$$

$\downarrow$

$$M = \sum_{ij} \psi_{ij} |i\rangle\langle j|$$

$$= \sum_{k=1}^{r} \sigma_k |u_k\rangle\langle v_k|$$

$$|\psi\rangle \stackrel{!}{=} \sum_{k=1}^{r} \sigma_k |u_k\rangle |v_k\rangle$$

Schmidt coefficients

"Schmidt vectors"

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$$A_0 A_1 \otimes I |\psi\rangle = - A_1 A_0 \otimes I |\psi\rangle$$

$$\sum_k \sigma_k (A_0 A_1 |u_k\rangle) \otimes |v_k\rangle$$

$$= \sum_k \sigma_k (- A_1 A_0 |u_k\rangle) \otimes |v_k\rangle$$

$\Longrightarrow \forall k \in \{1, \dots, r\}$

$$A_0 A_1 |u_k\rangle = -A_1 A_0 |u_k\rangle$$

Also rememb

$$A_0 \otimes I |\psi\rangle = I \otimes \frac{B_0 + B_1}{\sqrt{2}} |\psi\rangle$$

$$\sum \sigma_k (A_0 |u_k\rangle) |v_k\rangle = \sum \sigma_k \underline{\underline{|u_k\rangle}} \left(\frac{B_0+B_1}{\sqrt{2}} |v_k\rangle\right)$$

$$\forall k \quad \sigma_k A_0 |u_k\rangle = \sum_j \sigma_j |u_j\rangle \cdot \langle v_k | \frac{B_0+B_1}{\sqrt{2}} | v_j\rangle$$

$\Longrightarrow$ A_0, A_1 preserve $\mathrm{span}(\{u_1, \dots, u_k\})$

$A_0 A_1 = -A_1 A_0$ on span

$\Longrightarrow \exists\, U$ s.t.

$$U^\dagger A_0 U = \begin{pmatrix} Z \otimes I & 0 \\ 0 & A' \end{pmatrix}$$

$$\mathcal{U}^\dagger A_1 \mathcal{U} = \begin{pmatrix} X \otimes I & 0 \\ 0 & B' \end{pmatrix}$$

We've shown

If $S = (|\psi\rangle, A_0, A_1, B_0, B_1)$

is a perfect CHSH strat.

$\Longrightarrow$ up to change of basis

$$A_0 = \begin{pmatrix} Z \otimes I & 0 \\ 0 & \cdots \end{pmatrix} \quad A_1 = \begin{pmatrix} X \otimes I & 0 \\ 0 & \cdots \end{pmatrix}$$

$$\frac{B_0 + B_1}{\sqrt{2}} = \begin{pmatrix} Z \otimes I & 0 \\ 0 & \cdots \end{pmatrix} \quad \frac{B_0 - B_1}{\sqrt{2}} = \begin{pmatrix} X \otimes I & 0 \\ 0 & \cdots \end{pmatrix}$$

To characterize $|\psi\rangle$, use stabilizers

$$\langle \psi | CHSH | \psi \rangle = \sqrt{2} \left(\langle \psi | \begin{matrix} Z \\ \otimes \\ I \end{matrix} \otimes \begin{matrix} Z \\ \otimes \\ I \end{matrix} | \psi \rangle + \langle \psi | \begin{matrix} X \\ \otimes \\ I \end{matrix} \otimes \begin{matrix} X \\ \otimes \\ I \end{matrix} | \psi \rangle \right)$$

$$= 2\sqrt{2}$$

$\Rightarrow |\psi\rangle$ is stabilized by

$$\begin{array}{c} Z \\ \otimes \\ I \end{array} \otimes \begin{array}{c} Z \\ \otimes \\ I \end{array} \; ; \; \begin{array}{c} X \\ \otimes \\ I \end{array} \otimes \begin{array}{c} X \\ \otimes \\ I \end{array}$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |aux\rangle$$

$$= |EPR\rangle \otimes |aux\rangle$$

What we really need:

$$\langle\psi| CHSH |\psi\rangle \geq 2\sqrt{2} - \varepsilon$$

$$\Rightarrow |\psi\rangle \approx |EPR\rangle \otimes |aux\rangle$$

$$A_0 \approx Z \otimes I$$

$$\vdots$$

Self-testing rigidity

"robust"

$$\sqrt{2} \cdot CHSH = 4I - p^2 - q^2$$

$$\langle\psi|CHSH|\psi\rangle = 2\sqrt{2} - \varepsilon$$

$$\Rightarrow \langle\psi|(p^2+q^2)|\psi\rangle \le \varepsilon\sqrt{2}$$

$$\Rightarrow \langle\psi|p^2|\psi\rangle \le \varepsilon\sqrt{2}$$
$$\langle\psi|q^2|\psi\rangle \le \varepsilon\sqrt{2}$$

$$\Longleftrightarrow \quad \| p|\psi\rangle \|^2 \le \varepsilon\sqrt{2}$$

$$\left\|\left(A_0\otimes I - I\otimes\frac{B_0+B_1}{\sqrt{2}}\right)|\psi\rangle\right\|^2 \le \varepsilon\sqrt{2}$$

Abbreviate:

$$A_0\otimes I_{|\psi\rangle} \approx_{\varepsilon\sqrt{2}} I\otimes\frac{B_0+B_1}{\sqrt{2}}\,|\psi\rangle$$

Recall: In exact case, used this to deduce $A_0A_1\otimes I|\psi\rangle$
$= -A_1A_0\otimes I|\psi\rangle$

Still true that

$$(B_0+B_1)(B_0-B_1) = -(B_0-B_1)(B_0+B_1)$$

Show approximate anticom:

$$\underbrace{A_0A_1\otimes I}|\psi\rangle \approx A_0^{\otimes I}\left(I\otimes\frac{B_0-B_1}{\sqrt2}\right)|\psi\rangle$$

$$=\left(I\otimes\frac{B_0-B_1}{\sqrt2}\right)(A_0\otimes I)|\psi\rangle$$

$$\approx\left(I\otimes\frac{B_0-B_1}{\sqrt2}\right)\left(I\otimes\frac{B_0+B_1}{\sqrt2}\right)|\psi\rangle$$

$$=-\left(I\otimes\frac{B_0+B_1}{\sqrt2}\right)\left(I\otimes\frac{B_0-B_1}{\sqrt2}\right)|\psi\rangle$$

$$\approx-\left(I\otimes\frac{B_0+B_1}{\sqrt2}\right)(A_1\otimes I)|\psi\rangle$$

$$= -(A_1 \otimes I)\left(I \otimes \frac{B_0 + B_1}{\sqrt{2}}\right)|\psi\rangle$$

$$\hat{\approx} -(A_1 \otimes I)(A_0 \otimes I)|\psi\rangle$$

$$= -A_1 A_0 \otimes I|\psi\rangle$$

Note: can get optimal constants starting from pf. #2 of Tsirelson

$$A_0 A_1 \otimes I|\psi\rangle \approx_{O(\varepsilon)} -A_1 A_0 \otimes I|\psi\rangle$$

Step 2:

$\hat{\approx}$ anticom $\Longrightarrow$ $\hat{\approx}$ X & Z

Use "swap isometry"

also show $|\psi\rangle \hat{\approx} |EPR\rangle \otimes |aux\rangle$

Self-testing:

$A \approx \tilde{A} \leftarrow$ target/ideal

$|\psi\rangle \hat{\approx} |\tilde{\psi}\rangle \leftarrow$ ideal

$\exists$ local isometry V_A, V_B

s.t.

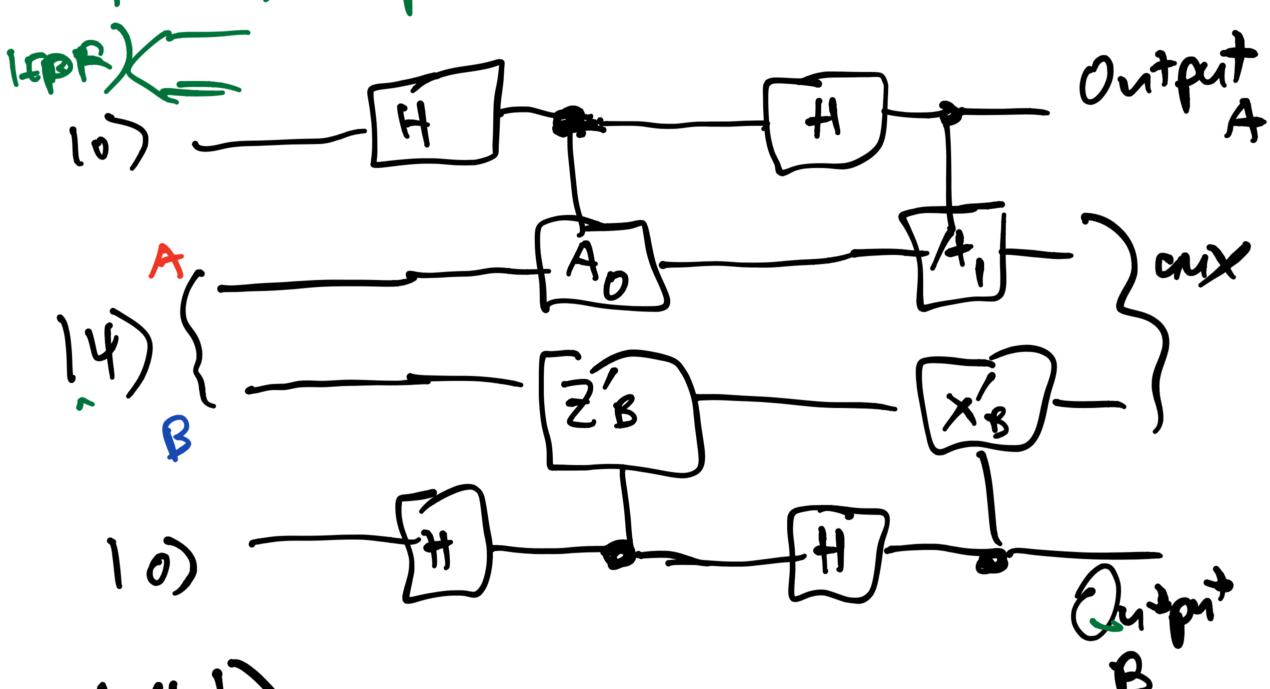
$$(V_A \otimes V_B)|\psi\rangle \approx |\tilde{\psi}\rangle \otimes |aux\rangle$$

$$(V_A \otimes V_B)(A \otimes I|\psi\rangle) \approx (\hat{A}|\tilde{\psi}\rangle) \otimes |aux\rangle$$

isometry: "rectangular unitary"

$$V^\dagger V = I$$

The swap isometry

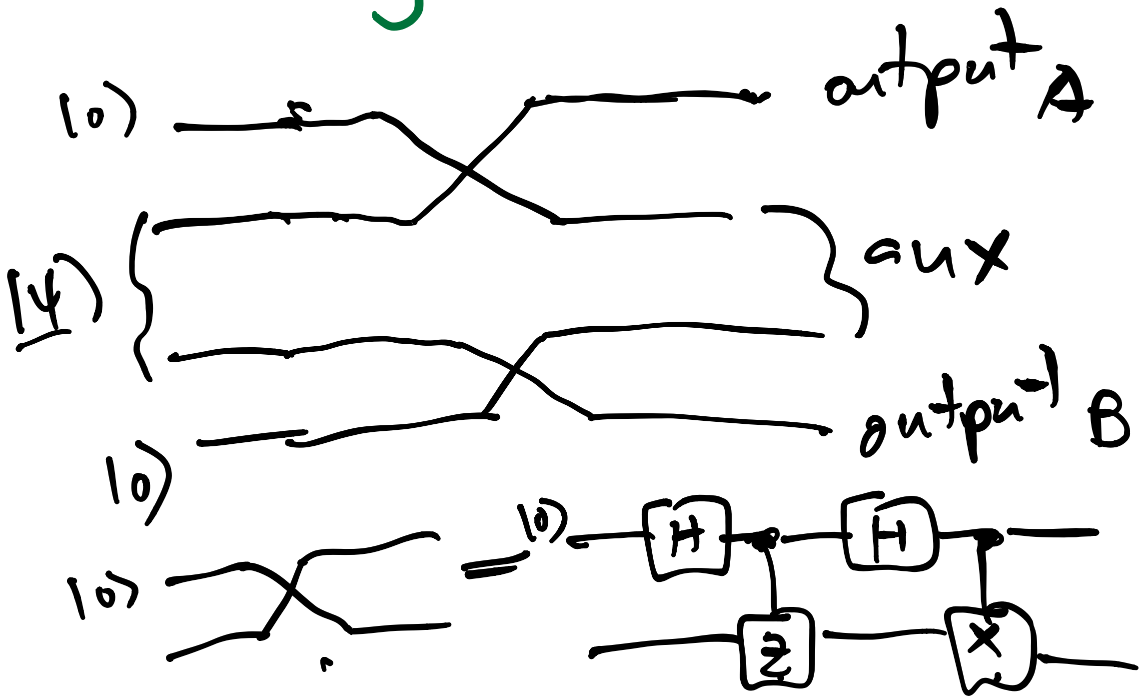


$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

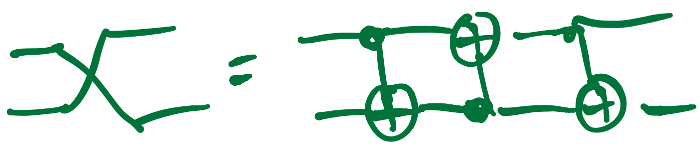
$$Z'_B = \text{reg}\left(\frac{B_0+B_1}{\sqrt{2}}\right)$$

$$X'_B = \text{reg}\left(\frac{B_0-B_1}{\sqrt{2}}\right)$$

"Morally"



Pf. hint:



In reality:

$$(V_A \otimes V_B)|\psi\rangle$$

$$= \frac{1}{4}\Big(|00\rangle \otimes (I + A_0)(I + Z'_B)|\psi\rangle$$

$$+ |01\rangle \otimes (I + A_0)X'_B(I - Z'_B)|\psi\rangle$$

$$+ |10\rangle \otimes A_1(I - A_0)(I + Z'_B)|\psi\rangle$$

$$+ |11\rangle \otimes A_1(I - A_0)X'_B(I - Z'_B)|\psi\rangle \Big)$$

small

$\approx$

We know:

$$A_0A_1 \otimes I|\psi\rangle \approx -A_1A_0 \otimes I|\psi\rangle$$

$$I \otimes Z'_B X'_B|\psi\rangle \approx -I \otimes X'_B Z'_B|\psi\rangle$$

$$A_0 \otimes I|\psi\rangle \approx I \otimes Z'_B|\psi\rangle$$

$$A_1 \otimes I|\psi\rangle \approx I \otimes X'_B|\psi\rangle$$

$$V_A \otimes V_B|\psi\rangle \approx \frac{1}{4}\Big(|00\rangle|aux\rangle + |11\rangle|aux\rangle\Big)$$

$$(V_A \otimes V_B)(A_0 \otimes I)|\psi\rangle$$

$$\approx \frac{1}{4}\Big(|00\rangle|aux\rangle - |11\rangle|aux\rangle$$
$$= (Z \otimes I)\left(\frac{1}{4}(|00\rangle + |11\rangle)|aux\rangle\right)\Big)$$

Error $\approx O(\varepsilon^{1/4})$

Next time: why is this useful?

Say I lay CHSH many times

observe success ≈ 0.854

$\Longrightarrow$ Alice & Bob's measurement outcomes are highly random

Certify quantum computations