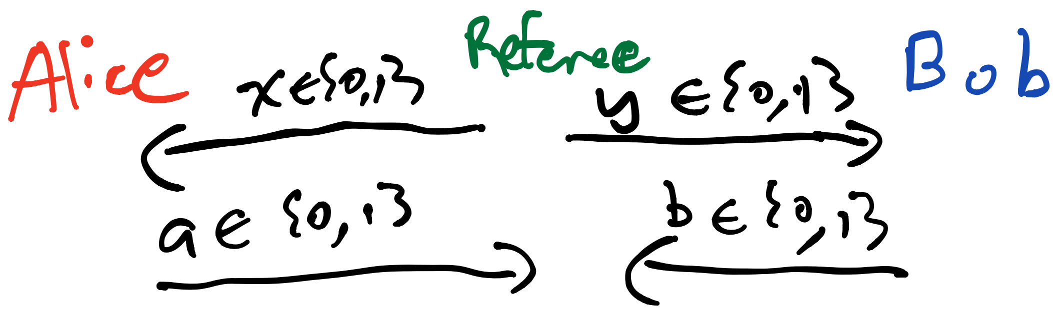
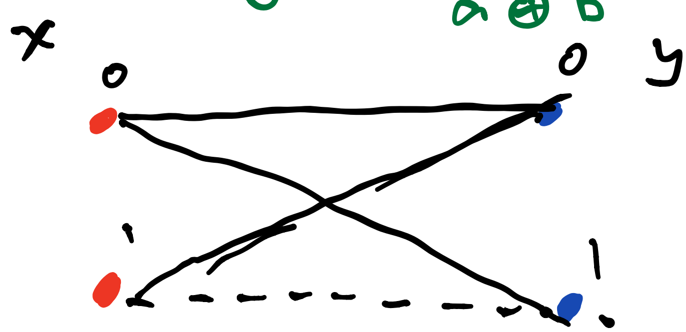
6.S979 Lecture 2: CHSH game

Bell '64   Clauser Horne Shimony Holt '69

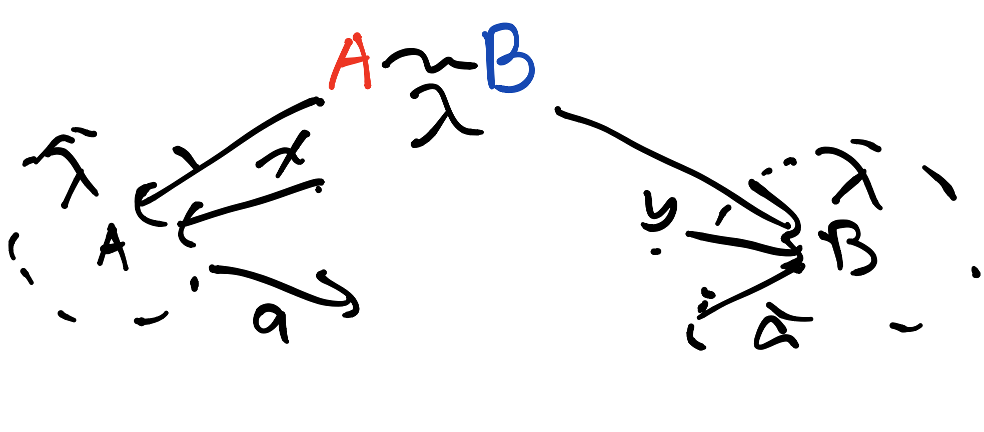


$$x \cdot y \stackrel{?}{=} a + b \mod 2$$

$$a \oplus b$$



LHVM ("Classical strategies")



$$\left.\begin{aligned} a &= f(\underline{\lambda}, x) \\ b &= g(\lambda, y) \end{aligned}\right\}$$

classical value

$$\omega = \sup \mathbb{E}_{\underline{\lambda} \sim \mu} \mathbb{E}_{\underline{x,y}} \mathbb{I}\left[xy = f(\lambda, x) + g(\lambda, y)\right]$$

λ doesn't matter!

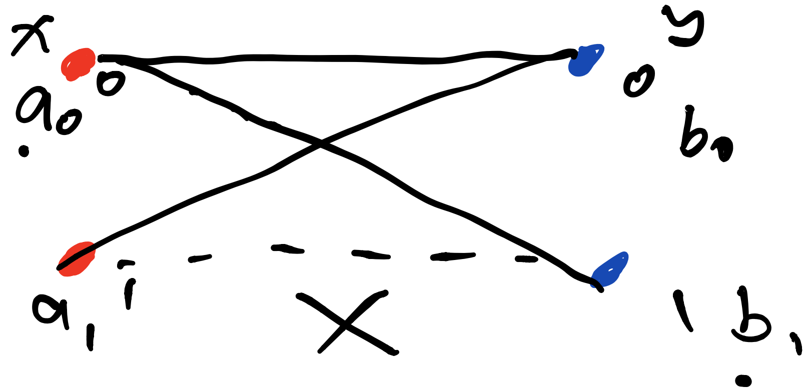
— Fix f, g. Then pick best λ

$$\omega = \sup_{\underline{f}, \underline{g}} \mathbb{E}_{x,y} \mathbb{I}\left[xy = f(x) + g(y)\right]$$

$a_0 = f(0)$, $a_1 = f(1)$

$b_0 = g(0)$, $b_1 = g(1)$

$$\omega = \sup_{\substack{a_0, b_0 \\ a_1, b_1}} \mathbb{E}_{x,y} \mathbb{I}\left[xy = a_x + b_y\right]$$



$$a_0 = b_0 = b_1 = a_1$$

$$a_1 \neq b_1$$

You can satisfy at most 3/4 constraints

$$\Longrightarrow \omega = 3/4$$

"CHSH inequality"

Violating CHSH w/ Quantum

Observables:

$a, b \in \{0, 1\}$

$A = (-1)^a$

$B = (-1)^b$

$$\mathbb{1}[xy = a+b]$$
$$= \frac{1}{2} + \frac{1}{2}(-1)^{xy} A B$$

$$\omega(S) = \frac{1}{2} + \underbrace{\frac{1}{2}\cdot\frac{1}{4}\left(A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1\right)}_{\text{bias}}$$

Binary measurement

$\Pi_0 \quad \Pi_1$

s.t. $\Pi_0^2 = \Pi_0, \; \Pi_1^2 = \Pi_1$

$\Pi_0 + \Pi_1 = I$

$\Pi_0 \cdot \Pi_1 = \Pi_1 \cdot \Pi_0 = 0$

$O = \Pi_0 - \Pi_1$

Hermitian, eigenvalue ± 1

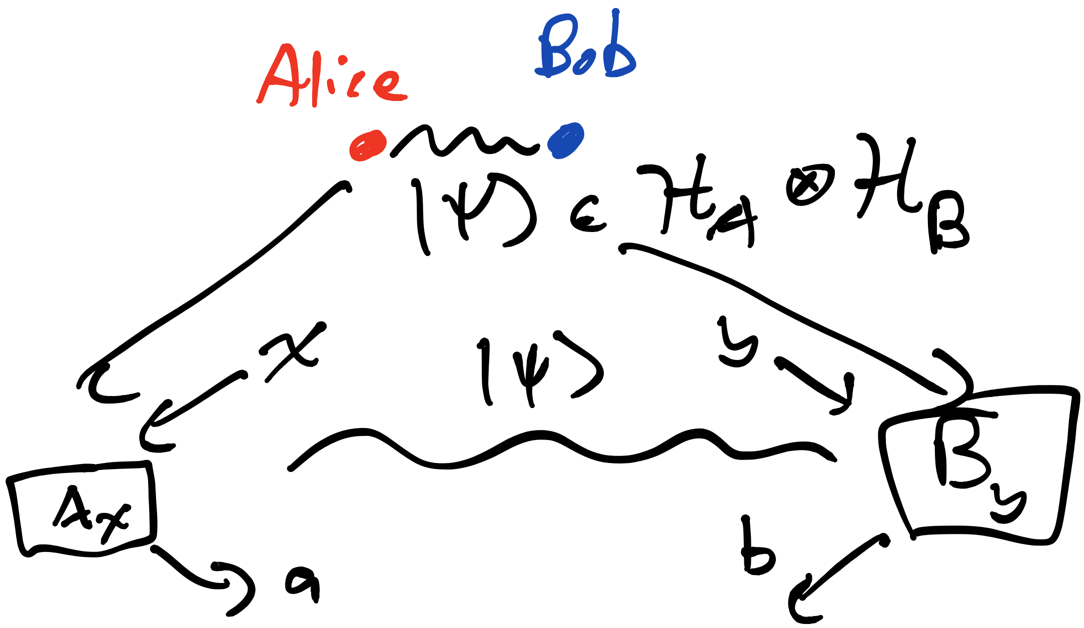
proj. on +1 eigenspace

proj. on −1 eigenspace

$$\langle \psi | \mathcal{O} | \psi \rangle$$

= Expected value in ±1 notation

$= 1 \cdot \Pr[0] - 1 \cdot \Pr[1]$



$$\omega^* = \sup_{\substack{|\psi\rangle \\ A_x, B_y}} \left(\frac{1}{2} + \frac{1}{8} \langle \psi | A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 | \psi \rangle \right)$$

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

Z-basis $\quad |0\rangle \quad |1\rangle$

X-basis $\quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$

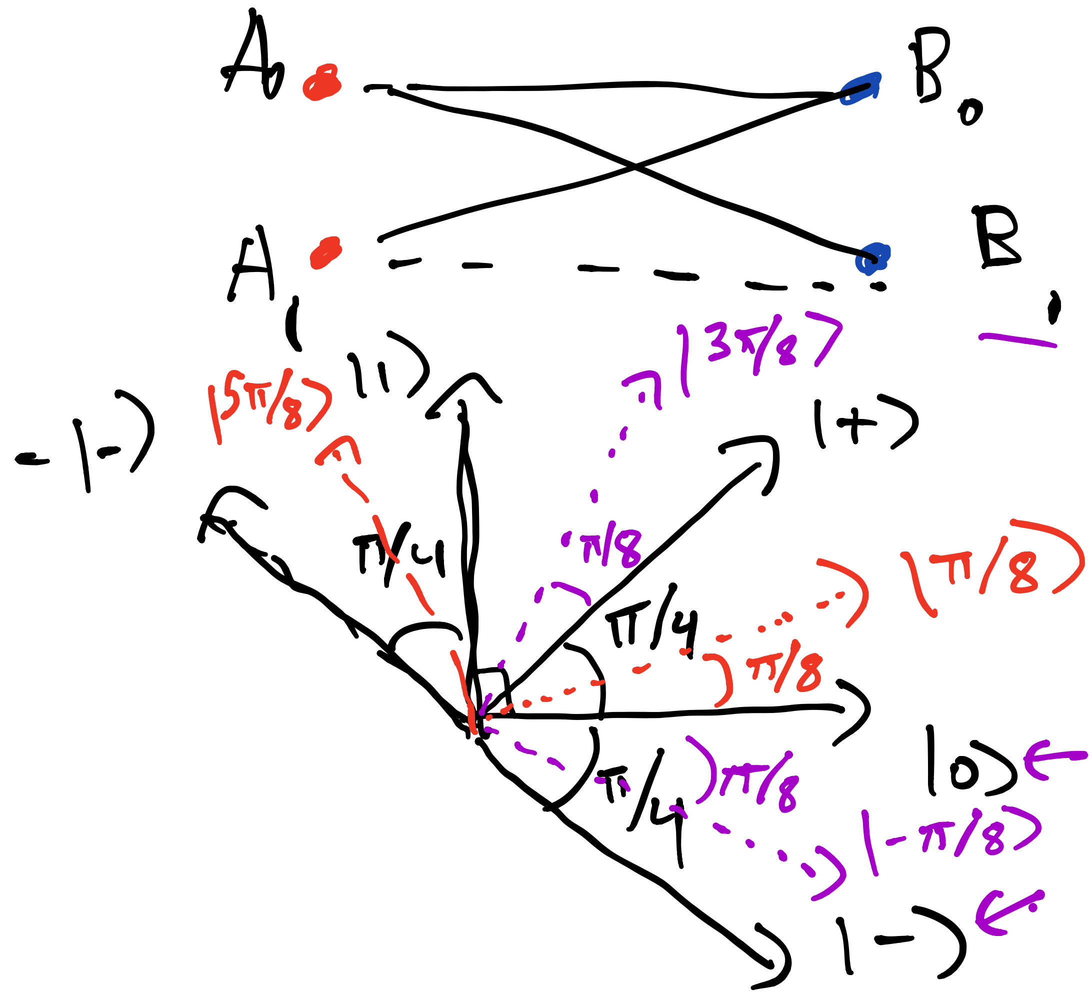
$$Z = \Pi_0 - \Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = \underset{|+\rangle\langle+|}{\Pi_+} - \Pi_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A_0 = Z, \quad A_1 = X$$



$A_0 = Z, \quad A_1 = X$

$$B_0 = |\pi/8\rangle\langle\pi/8| - |5\pi/8\rangle\langle 5\pi/8|$$

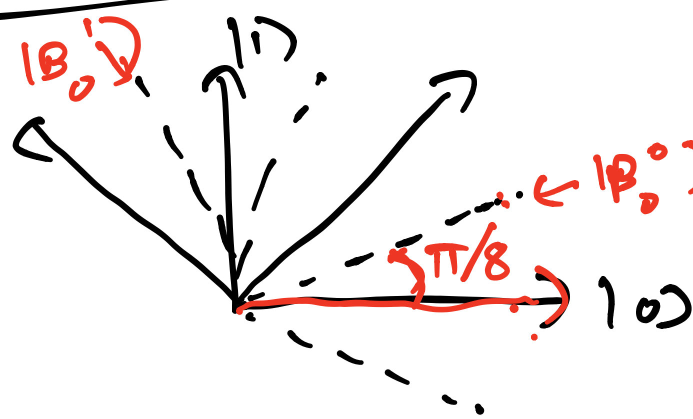
$$|\pi/8\rangle = \cos(\pi/8)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle$$

$$|5\pi/8\rangle = -\sin(\pi/8)|0\rangle + \cos\left(\frac{\pi}{8}\right)|1\rangle$$

$$B_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$B_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

Calculating win prob:



$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$x = 0$ $\quad y = 0$

A_0 $\quad B_0$

$= Z$

$a = 0$ or 1 w/ prob. $1/2$

$|00\rangle$ $\quad |11\rangle \rightarrow$ $b = 1$ w/ prob. $\cos^2\frac{\pi}{8}$

$\qquad 0$ w/ prob. $\sin^2\frac{\pi}{8}$

$b = 0$ w/ prob. $\cos^2\left(\frac{\pi}{8}\right)$

$\qquad 1$ w/ prob. $\sin^2(\pi/8)$

$a = b$ w/ prob. $\cos^2\frac{\pi}{8}$

$a \neq b$ w/ prob $\cos^2\frac{\pi}{8}$

$\Rightarrow$ win prob. $\cos^2\frac{\pi}{8}$

≈ 0.854

Stabilizer calculation:

"stabilizers of $|\psi\rangle$"

Observe:

$$|\psi\rangle = X\otimes X|\psi\rangle \checkmark$$

$$\overset{\shortparallel}{\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)} = Z\otimes Z|\psi\rangle \checkmark$$

$$X=\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

"bit flip" $X|0\rangle=|1\rangle$, $X|1\rangle=|0\rangle$

"phase flip" $Z|0\rangle=|0\rangle$, $Z|1\rangle=-|1\rangle$

$$B_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{\sqrt{2}}(Z+X)$$

$$B_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix} = \frac{1}{\sqrt{2}}(Z-X)$$

$$\omega^*(S) = \frac{1}{2} + \frac{1}{8}\langle\psi|\big(A_0\otimes B_0 + A_0 \otimes B_1 + A_1\otimes B_0 - A_1 \otimes B_1\big)|\psi\rangle$$

$$= \frac{1}{2} + \frac{1}{8}\cdot\frac{1}{\sqrt{2}}\langle\psi|\big(Z\otimes(Z+X) + Z\otimes(Z-X) + X\otimes(Z+X) - X\otimes(Z-X)\big)|\psi\rangle$$

$$= \frac{1}{2} + \frac{1}{8}\cdot\frac{1}{\sqrt{2}}\langle\psi|\big(2Z\otimes Z + 2X\otimes X\big)|\psi\rangle$$

$$= \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\langle\psi|\big(Z\otimes Z + X\otimes X\big)|\psi\rangle$$

$$= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\langle\psi|\psi\rangle$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2\left(\frac{\pi}{8}\right) \approx 0.854$$

What does this mean?

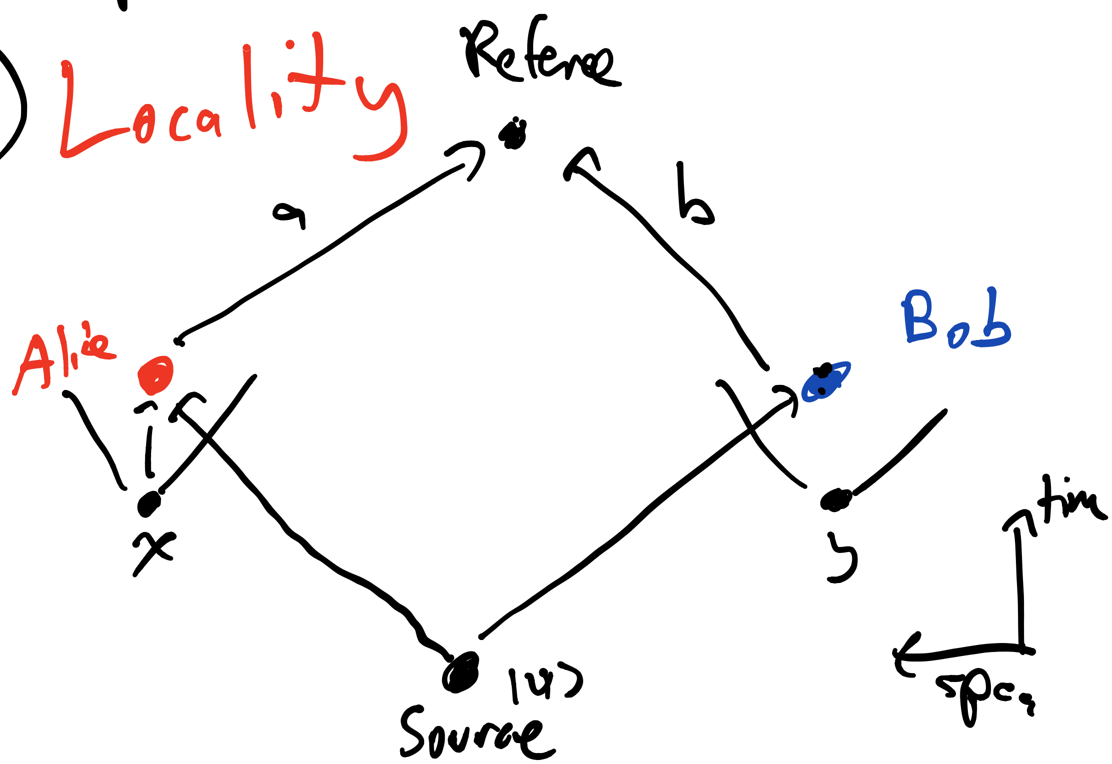
If A & B play CHSH

win w/ prob. $> 3/4$

$\Longrightarrow$ no LHVM for their systems

Assumptions

1) Locality



2) Repeated trials / memory ✓

3) Detection loophole
biased sample of the rounds

Aspect '80s

Hensen '15
+ others

Why 0.854 ???

Is it optimal? Yes in QM (Tsirelson bound)

Could you get $\omega^* = 1$?

$$\langle\psi|(\overbrace{A_0\otimes B_0}^{+1} + \overbrace{A_0\otimes B_1}^{+1} + \overbrace{A_1\otimes B_0}^{+1} - \overbrace{A_1\otimes B_1}^{-1})|\psi\rangle$$

$$= 4$$

$$\underbrace{\langle\psi|(A_0\otimes I)}_{\langle v_0|}\cdot\underbrace{(I\otimes B_0)|\psi\rangle}_{|w_0\rangle} = 1$$

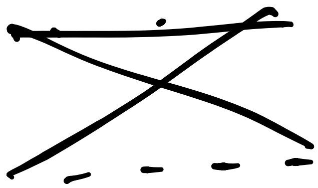
$$\langle v_0| \cdot |w_0\rangle = 1$$

$$|v_0\rangle = A_0^\dagger\otimes I|\psi\rangle \qquad |v_0\rangle$$

$$\Rightarrow |v_0\rangle = |w_0\rangle$$

$$|v_0\rangle = \alpha|w_0\rangle$$

$$\langle v_0|w_0\rangle = \alpha^*$$



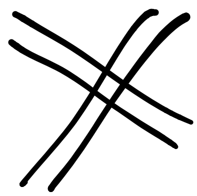
$|v_0\rangle = |w_0\rangle$

$|v_0\rangle = |w_1\rangle$

$|v_1\rangle = |w_0\rangle$

$\langle v_1 | w_1\rangle = -1$

$|v_1\rangle = -|w_1\rangle$



no perfect q. <u>strat</u>