6.S979 Lecture 10

- Next class on 10/13 at this time

Last time:

XOR correlations

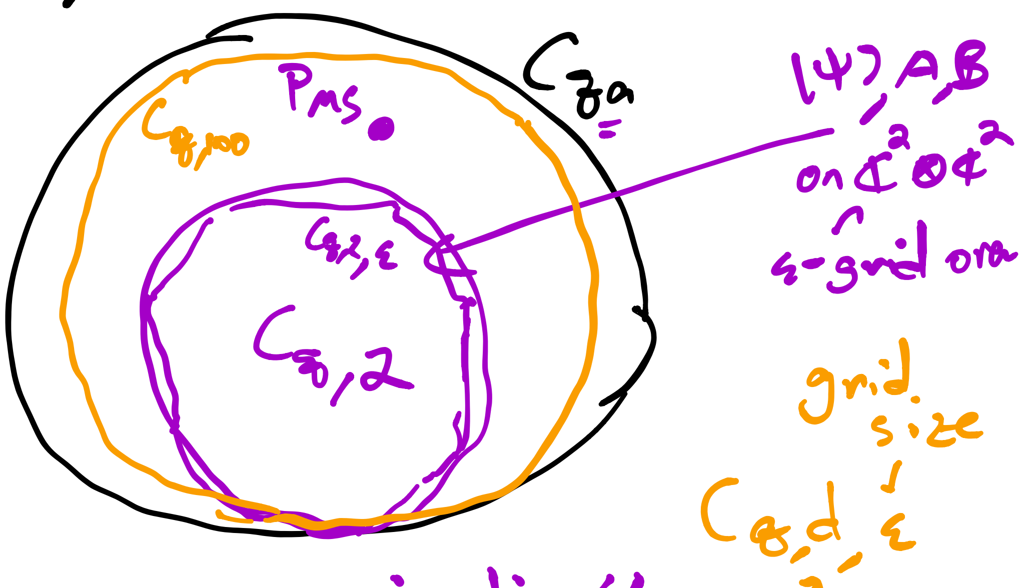
$$C_q^{\oplus} = C_{qs}^{\oplus} = C_{qa}^{\oplus} = C_{qc}^{\oplus} = C_{\text{Tsirelson}}^{\oplus}$$

$$C_{\text{Tsirelson}}^{\oplus} = \left\{ C_{xy} : \begin{pmatrix} 1 & & C_{xy} \\ & 1 & \\ C_{xy} & & 1 \\ & & & 1 \end{pmatrix} \succeq 0 \right\}$$

$$\{ C_{xy} = \langle \vec{u}_x, \vec{u}_y \rangle \}$$

C_{qa} = "quantum approximate"

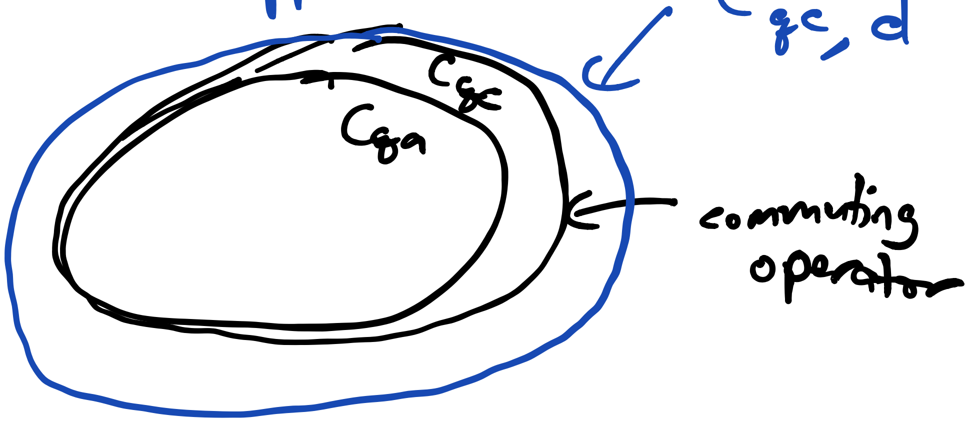
$$p(a,b|x,y) \approx_\varepsilon \langle\psi| A^x_a \otimes B^y_b |\psi\rangle$$



"inner approximation"

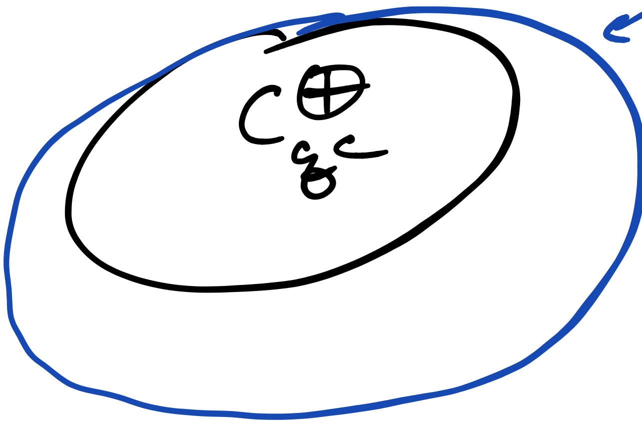
$$\lim_{d\to\infty} \lim_{\varepsilon\to 0} C_{q,d,\varepsilon} = C_{qa} = \overline{C_q}$$

"outer approximation"



$$\lim_{d\to\infty} C_{qc,d} \to C_{qc}$$

$$\{C_{xy} : |c_{xy}| \le 1\}$$



For XOR

$$C_{ij} = \langle\psi| P_i P_j |\psi\rangle$$

observables (A_x or B_y)

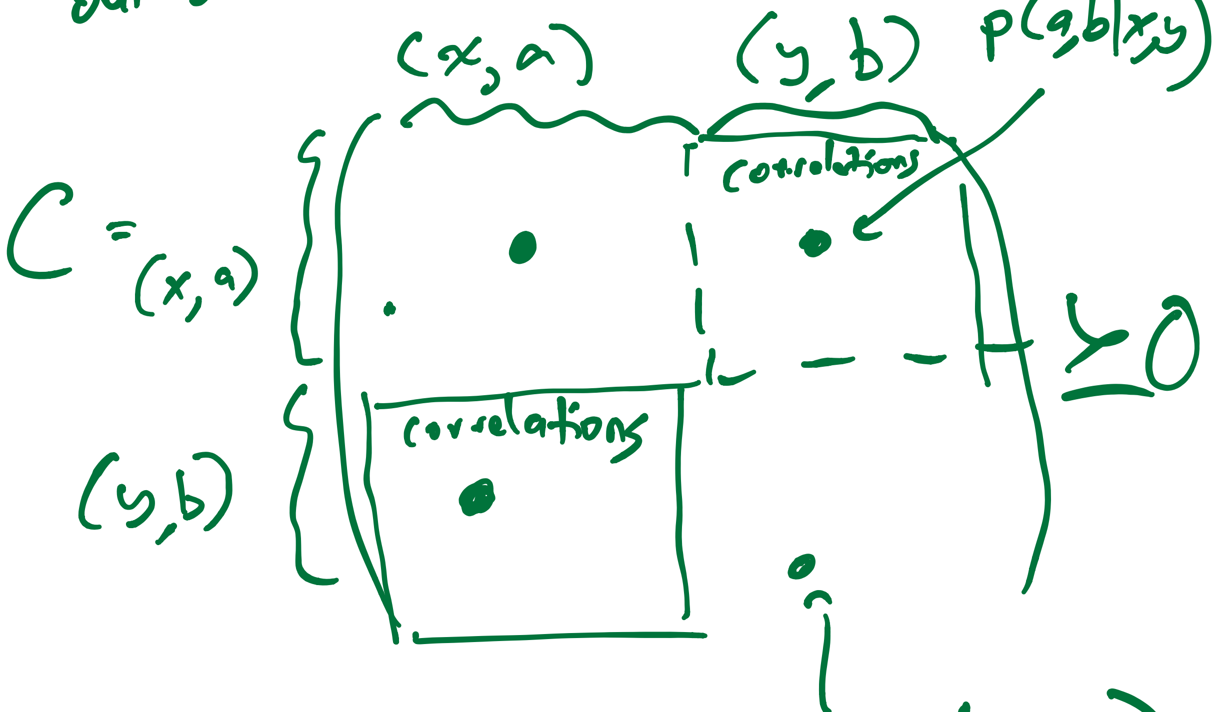
$$= \langle \vec{u}_i, \vec{u}_j \rangle$$

$$P(a,b|x,y) = \langle\psi| A^x_a B^y_b |\psi\rangle$$

$$(A^x_a)^2 = A^x_a, \quad \sum_a A^x_a = I$$

$$[A^x_a, B^y_b] = 0$$

Try to make a Gram matrix out of these vectors



$$C_{(y,b),(y',b')} = \langle\psi| B^y_b B^{y'}_{b'} |\psi\rangle \in \mathbb{C}$$

$$C_{(x,a)(x,a)} = \langle\psi|(A^x_a)^2|\psi\rangle$$
$$= \langle\psi| A^x_a |\psi\rangle$$

Fix x vary a

$$\langle \psi | \cancel{A^x_a A^x_{a'}} | \psi \rangle \to 0$$

$$\mathrm{tr}\left(\begin{array}{c} a_1 \\ \vdots \\ a_k \end{array} \overbrace{\begin{pmatrix} \ddots & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & \ddots \end{pmatrix}}^{a_1 \dots a_k} \right) = I$$

$$\sum_a \langle \psi | (A^x_a)^2 | \psi \rangle$$

$$= \sum_a \langle \psi | A^x_a | \psi \rangle$$

$$= \langle \psi | I | \psi \rangle$$

$$= 1$$

These give you necessary conditions on a correlation, but not sufficient

$$C \implies \text{Gram vectors } \vec{w}_{x,a} \ \vec{w}_{y,b} \ \cancel{\implies} A^x_a \ B^y_b$$

$$\langle \psi | A^x_a | \psi \rangle$$

Gram vector $|\psi\rangle$

$$C = \begin{array}{c} \\ \emptyset \\ (x,a) \\ (y,b) \end{array} \begin{pmatrix} I & \overbrace{\langle \psi | A^x_a | \psi \rangle}^{(x,a)} & \overbrace{\langle \psi | B^y_b | \psi \rangle}^{(y,b)} \\ \langle \psi | A^x_a | \psi \rangle & \text{As above} & \\ \langle \psi | B^y_b | \psi \rangle & & \end{pmatrix} \geq 0$$

(y,b)

$A^x_a \quad B^y_b$ rows & cols of C are all products of degree ≤ 1 in these operators

What about higher degree?

$$C_{(x,a)(x',a')\,|\,(y,b)} = \underbrace{\langle\psi| A^x_a A^{x'}_{a'}}\ \underbrace{B^y_b|\psi\rangle}$$

$\langle\psi|A^x_a B^y_b|A^{x'}_{a'}|\psi\rangle = C_{(x,a)(y,b)\,(x',a')}$

$$C = \begin{array}{c} \\ \emptyset \\ (x,a) \\ (y,b) \\ (x,a)(x',a') \\ (x,a)(y,b) \\ (y,b)(y',b') \end{array} \begin{array}{c} \emptyset \quad \cdots \\ \left(\begin{array}{ccc} & & \\ & & \\ \bullet & \xleftrightarrow{\text{equal}} & \bullet \\ & & \end{array}\right) \end{array} \succeq 0$$

For any degree d, you can

write down an "expanded correlation matrix" w/ rows, cols labeled by <u>monomials</u> in A^x_a, B^y_b of degree $\leq d$

$$C^{(d)} \succeq 0$$

"psd constraint"

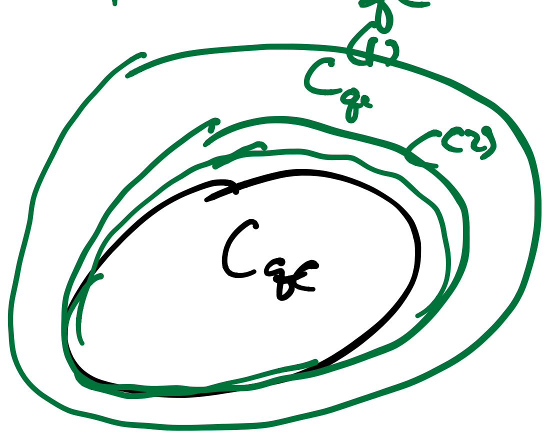
$$\mathrm{tr}\, C^{(d)}_{\text{block}} = I$$

$$C^{(d)}_{i,j} = C^{(d)}_{i',j'}$$

$\vdots$

"linear constraints"

For any d, the set of $C^{(d)}$ matrices is an outer approx to C_{qc}



Thm: [Navascués, Pironio, Acín '08]

$$\lim_{d\to\infty} C_{qa}^{(d)} = C_{qc}$$

Compare: $C_{qc}^{\oplus,(1)} = C_{qc}^{\oplus}$

"NPA hierarchy"

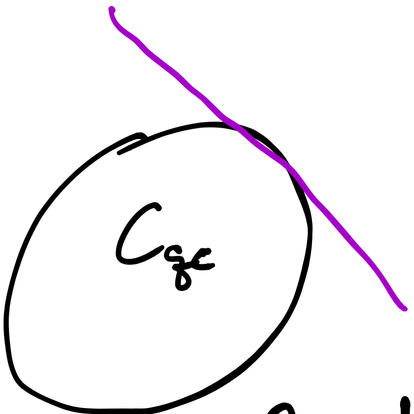
Level d can be computed in time 2^d

"Dual picture"

Recall that a game / Bell inequality is a hyperplane in the space of correlations

$$\omega^* \geq \sum_{x,y,a,b} \underline{\pi}(x,y) \cdot p(a,b|x,y) \cdot C_{a,b|x,y}$$

$$= \langle \vec{v}, \vec{p} \rangle$$



Recall proof #3 of Tsirelson

$$2\sqrt{2}\cdot I - \underbrace{CHSH}_{\substack{A_0B_0 + A_0B_1 \\ + A_1B_0 - A_1B_1}} = \frac{1}{\sqrt{2}}\left(A_0 - \frac{B_0+B_1}{\sqrt{2}}\right)^2 + \frac{1}{\sqrt{2}}\left(A_1 - \frac{B_0-B_1}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad CHSH \preceq 2\sqrt{2}\cdot I$$

$$\Rightarrow \langle \psi | CHSH | \psi \rangle \leq 2\sqrt{2}$$

$$\Rightarrow p_{win} \leq \frac{1}{2} + \frac{2\sqrt{2}}{8}$$

$$\left.\begin{aligned}2\sqrt{2}I - CHSH = {} & \frac{1}{\sqrt{2}}\left(A_0 - \frac{B_0+B_1}{\sqrt{2}}\right)^2 \\ & + \frac{1}{\sqrt{2}}\left(A_1 - \frac{B_0-B_1}{\sqrt{2}}\right)^2\end{aligned}\right\}\ \text{SOS terms}$$

$$\text{"constraint terms"}\ \left\{\begin{aligned} & - \frac{1}{\sqrt{2}}\Big((A_0^2 - I) + (A_1^2 - I) \\ & \qquad + (B_0^2 - I) + (B_1^2 - I)\Big) \\ & - \frac{1}{2}\Big([A_0, B_0] + [A_0, B_1] \\ & \qquad + [A_1, B_0] + [A_1, B_1]\Big)\end{aligned}\right.$$

$$2\sqrt{2}\, I - CHSH = P_1^2 + P_2^2 \quad \text{mod constraints}$$

$$\left\{ (A_i^2 - I),\ [A_i, B_j] \right\}$$

Generalization: say I have a game

$\pi(x,y)$ $\quad G_{a,b|x,y}$

Define "game polynomial"

$$g(A, B) = \sum_{x,y,a,b} \pi(x,y) \cdot C_{a,b|x,y} \cdot A_a^x \, B_b^y$$

(e.g., $g_{CHSH} = \frac{1}{4} + \frac{1}{8}(A_0 B_0 + \ldots))$

$$\alpha \cdot I - g \succeq 0$$

$\forall$ $\underline{A}$, $\underline{B}$ that form valid measurements

$\Updownarrow$

$$\langle\psi|g|\psi\rangle \leq \alpha$$

$\shortparallel$

P_{win}

$\Uparrow$

$$\overbrace{\alpha \cdot I - g = \overbrace{\sum_i \underline{\underline{r_i}}^\dagger \underline{r_i}}^{\text{SOS terms}} + \underbrace{\sum_{ij} \left(\underline{\underline{f_{ij}}}\, \underline{p_i} + p_i f_{ij}^\dagger\right)}_{\text{constraint terms}}}^{\text{SOS certificate}}$$

$p_i = 0$ for valid measurements

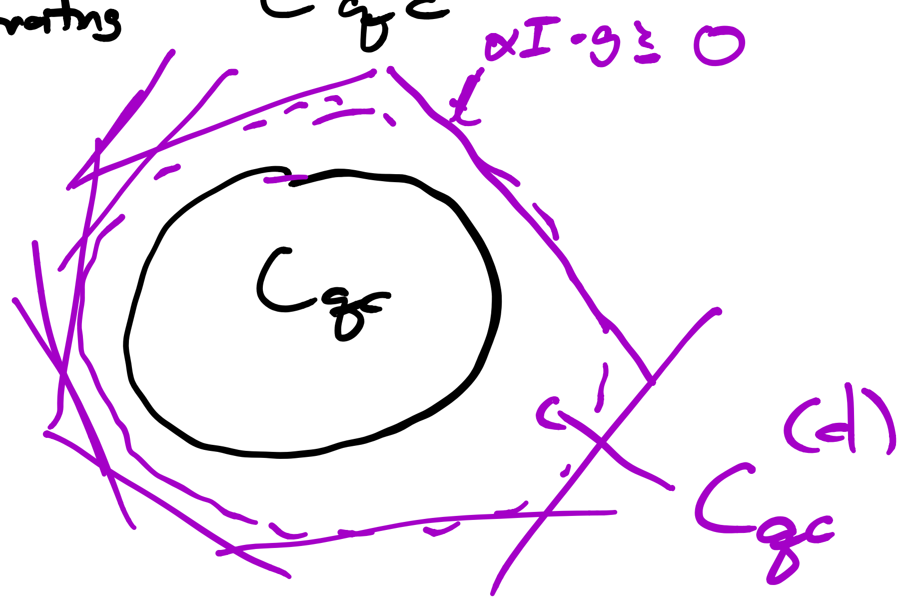
e.g. $\left(\sum A_a^x - I\right)$

$_a$

$$(A^x_a)^2 - A^x_a$$

$$[A^x_a, B^y_b]$$

Look at all SoS certificates
of degree $\leq d$
gives you a set of hyperplanes
separating C_{qc}



$C_{qc}^{(d)}$ is defined by degree-d
SoS certificates
SDP duality

Thm [non-commutative Positivstellensatz]

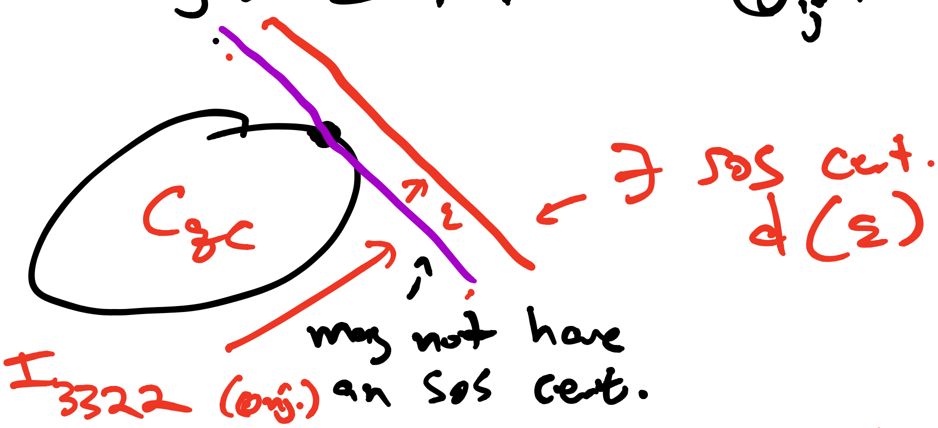
→ Helton McCullough ≈ '08

→ Doherty Liang Toner Wehner '08

for any $f > 0$ for A, B valid measurements

$\exists$ SOS certificate of this

$$f = \sum r_i^\dagger r_i + \sum (g_j p_i + p_i g_j^\dagger)$$



$\Rightarrow$ $C_{qc}^{(d)}$ not necessarily $= C_{qc}$ for any finite d

Ex: Tilted CHSH inequality

$$\langle\psi|\,\underbrace{\alpha A_0 + A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1}_{\mathrm{CHSH}_\alpha}\,|\psi\rangle$$

$$\beta_\alpha I - \mathrm{CHSH}_\alpha \succeq 0$$

$$\beta(\alpha) = \dots \quad \mathrm{SOS}_\alpha$$