2.1 Overview

The dynamics of a closed quantum system is unitary. In other words, if initially the state of a quantum system is $|\psi\rangle$ then in a closed system this can evolve to $U|\psi\rangle$ for some unitary matrix U. In terms of density matrices this is equivalent to the map $|\psi\rangle\langle\psi| \mapsto U|\psi\rangle\langle\psi|U^\dagger$. It is known that quantum mechanics never creates or destroys information, but this is only true for closed systems.

2.2 Three Definitions of Quantum Operations

Today we discuss how to generalize quantum operation beyond unitaries. Along with unitary evolution, in general we can add or discard a quantum system

- **Add a system**: $\rho \mapsto \rho \otimes \sigma$
- **Discard a subsystem**: $\rho_{AB} \mapsto \rho_A = \mathrm{tr}_B(\rho_{AB})$

The latter operation is also known as partial trace:

$$\rho_A = \sum_b (I_A \otimes \langle b|)\rho_{AB}(I_A \otimes |b\rangle) = (\mathrm{id} \otimes \mathrm{tr}_B)\rho_{AB}$$

Notation: we need the following notation. If A and B are vector spaces, then $L(A), L(B)$ are the space of linear operations on A and B, and moreover $L(A, B)$ is the space of all linear operations : $A \to B$. We can also define $L(L(A), L(B))$ as the space of linear operations : $L(A) \to L(B)$. For example the mentioned partial trace belongs to $L(L(A \otimes B), L(A))$. In physics we refer to $L(L(A), L(B))$ as the space of super-operators.

2.2.1 First Form (Stinespring)

We claim that the only source of non-unitarity is adding or discarding quantum systems. In particular, any quantum operation can be described by some composition of (1) unitaries (2) adding a system (3) partial trace.

An isometry V is a map in $L(\mathbb{C}^{d_A}, \mathbb{C}^{d_B})$ for $d_B \geq d_A$ such that $V^\dagger V = I_{d_A}$. Note for any sate $|\psi\rangle$, $\| |\psi\rangle \| = \|V|\psi\rangle\|, \forall |\psi\rangle \in A$. Furthermore $VV^\dagger = I_B$ iff $d_A = d_B$. A key example of an isometry is the map $V|\psi\rangle = |\psi\rangle \otimes |0\rangle$. In terms of density matrices an isometry maps $\rho \mapsto V\rho V^\dagger$. For example the previous example adds $V\rho V^\dagger = \rho \otimes |0\rangle\langle 0|$.

We can combine unitarity and adding quantum systems with isometries. To complete the picture, we have:

- **Isometries**: $\mathcal{N}(\rho) = V\rho V^\dagger$, where V is an isometry, including unitaries, adding pure ancillas, etc

- **Partial Trace**: $\mathcal{N}(\rho_{AB}) = \mathrm{tr}_B(\rho_{AB})$

The next simplest thing we can do after the two primitive operations above is combine them: we apply an isometry and then take a partial trace.

$$\mathcal{N}(\rho) = \mathrm{tr}_E(V\rho V^\dagger) \qquad \textbf{(Form 1)}$$

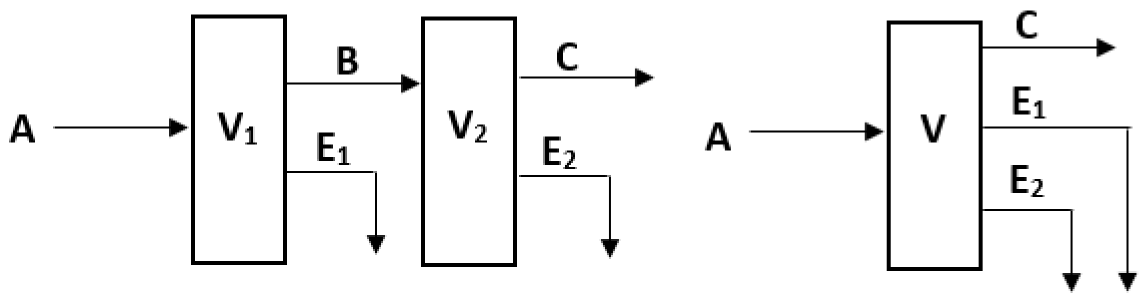
Here, V maps a system in A on to the space $B \otimes E$, and the partial trace throws out E, which could be the environment.

Even though this is simple, we claim it is fully general. This will be further justified later.

Another important thing to notice about this is that it is a composable operation. The composition of two isometries is another isometry, and taking the partial trace over two systems, E_1 and E_2, is just like taking the partial trace over some larger system. So if we can justify rearranging the operations, we can describe two quantum operations in **Form 1** as a single operation in **Form 1**.

Church of the Larger Hilbert Space: Whenever you do a partial trace, you can always do it later. Trace just means I commit to never looking at it again. It is all the same to do it now or later.

So we have the equivalence of the following two operations:



2.2.2 Second Form (Kraus operators)

From this perspective, we have another method for looking at channels. We can write isometries as $V = \sum_e V_e \otimes |e\rangle$, where $\{|e\rangle\}$ is an orthonormal basis and each $V_e \in L(A, B)$. This allows **Form 1** to be written as:

$$\mathcal{N}(\rho) = \mathrm{tr}_E\big[\sum_{e_1,e_2} V_{e_1}\rho V_{e_2}^\dagger \otimes |e_1\rangle\langle e_2|\big]$$

Evaluating the partial trace yields:

$$\mathcal{N}(\rho) = \sum_e V_e \rho V_e^\dagger \qquad \textbf{(Form 2)}$$

This allows us to describe the channel with no reference to the environment. **Form 2** is called the Kraus Decomposition, where V_e are the Kraus Operators of the channel and V is still an isometry.

What V_e are allowed here? We know their size from the fact that $V_e \in L(A, B)$. We also know they are blocks from a larger matrix, and that larger matrix is an isometry.

Remember the isometry condition $\mathbb{I} = V^\dagger V$. This can be expressed as a condition on all of the blocks by writing it out:

$$\mathbb{I} = (\sum_{e_1} V_{e_1}^\dagger \otimes \langle e_1|)(\sum_{e_2} V_{e_2} \otimes |e_2\rangle)$$

This gives us the Kraus Operator Conditions:

$$\sum_e V_e^\dagger V_e = \mathbb{I} \tag{2.1}$$

It is important to note that these are each positive semidefinite operators, so you don't get any cancellation.

This also works in reverse: if a set of $\{V_e\}$ satisfy the Kraus Operator Conditions, then the matrix V satisfies the isometry condition. That is, if we have a channel in form **Form 2**, it can also be written in **Form 1**. These forms are equivalent.

Here we give some concrete examples of Kraus operators

1. $\mathcal{N}(\rho) = U\rho U^\dagger$ with a single Kraus operator U.
2. $\mathcal{N}(\rho) = \sum_e p_e U_e \rho U_e^\dagger$ with the set of Kraus operators $\{\sqrt{p_e} U_e\}_e$.
3. $\mathcal{N}(\rho_{AB}) = \text{tr}\rho_{AB}$ with the set of Kraus operators $\{V_b = I_A \otimes \langle b|\}_b$.
4. **Cooling:** $\mathcal{N}(\rho) = |0\rangle\langle 0|$ for qubit input. The set of Kraus operators is $\{|0\rangle\langle 0|, |0\rangle\langle 1|\}$.
5. **Noise Map:** A bit flip with probability p. $\mathcal{N}(\rho) = (1-p)\rho + pX\rho X$. The Kraus operators are simply $(\sqrt{1-p})\mathbb{I}$ and $\sqrt{p}X$.
6. **Phase Damping**: $\mathcal{N}(\rho) = (1-p)\rho + pZ\rho Z$, with similar Kruas operators.
7. **Depolarizing Channel:** $\mathcal{N}(\rho) = (1-p)\rho + \frac{p}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) = (1-p)\rho + \frac{p}{2}\mathbb{I}$. In this channel, with probability p, the state is thrown away and replaced with the maximally mixed state.
8. Amplitude damping (problem set)

Note for the partial trace (item 3) we have the normalization $\sum_b V_b^\dagger V_b = \sum_b I_A \otimes |b\rangle_B \langle b| = I_A \otimes I_B$.

As a practical note, in a lot of physical systems that represent qubits, if we define it in terms of energy level, having a phase error is more likely than a bit flip. Maximal phase damping occurs when $p = \frac{1}{2}$ and this corresponds to measurement, which we will now discuss.

2.2.3 Third Form: axiomatic approach

The previous two forms have given models of quantum channels, but we have not yet shown that they can represent any general quantum operation. For this form, we will start from another perspective by asking "what properties should a general quantum operator satisfy?"

0. Hermeticity Preserving - A hermitian input should lead to a hermitian output.

1. Trace Preserving - Just as unitaries preserve length, our quantum operations should preserve trace.
2. Completely positive - Just like the nonnegativity condition on stochastic maps. Positive means that if ρ is nonnegative, then $\mathcal{N}(\rho)$ is nonnegative. However, we need a stronger condition for it to be correct. We need to stipulate that if we act on any part of ρ it should stay positive. That is if ρ_{AR} is positive semidefinite, then $(\mathcal{N} \otimes \mathrm{id}_R)(\rho)$ should also be positive semidefinite.
   (As a comparison, transpose is positive but not completely positive, as a side note: the partial positive transpose test (PPT) is one test of an entangled state: if PPT fails, it must be an entangled state).

These conditions give us the third form of quantum channels: a Trace Preserving, Completely Positive (TPCP) map. Since this is derived from the axioms of what a quantum channel should have, it describes the most general possible channel.

In fact, this turns out to be equivalent to the Kraus Form. We will only show that quantum channels in the Kraus Form are TPCP channels, and the converse will be left for the problem set. By taking the dagger of the Kraus condition, it is clear that it is hermeticity preserving. Looking at the trace $\mathrm{tr}(\mathcal{N}(\rho)) = \mathrm{tr}(\sum_e V_e \rho V_e^\dagger)$ and using the cyclic property of the trace and the Kraus Operator Condition, it can be seen that it is trace-preserving. Finally, we show that it is completely positive. If ρ is positive semidefinite, it can be written as $WW^\dagger$. Then $(\mathcal{N} \otimes \mathbb{I})\rho = \sum_e (V_e \otimes \mathbb{I}) W W^\dagger (V_e^\dagger \otimes \mathbb{I})$. This is a sum of positive semidefinite things, operators times their adjoints, and thus itself is positive semidefinite.

2.3 Measurement as a Quantum Operation

Measurement can be thought of as a quantum operation where the input is any quantum state and the output is classical, $\mathcal{N}(\rho) = \sum_x p_x |x\rangle\langle x|$ (diagonal density matrix). The probabilities, p_x, should depend on the state. Further, the p_x should be a linear function of $\rho = \mathrm{tr}(\rho M_x)$. From this we can work out the properties that M_x should obey:

1. Normalization: $\mathrm{tr}(\sum_x M_x \rho) = 1$ meaning $\sum_x M_x = \mathbb{I}$.
2. Positive Semidefinite: $\mathrm{tr}(M_x \rho) > 0$ and ρ is positive semidefinte.

These are similar to the Kraus Operator Conditions.

Just as we discussed the most general possible quantum operators, these are the most general measurements:

$$\sum_x M_x = I$$

$$M_x \succeq 0$$

These conditions still leave room for noisy measurements, etc.

We can also talk about non-demolition measurements, which do not discard the quantum systems after measurements. Consider the following quantum channel

$$\mathcal{N}(\rho) = \sum_e V_e \rho V_e^\dagger \otimes |e\rangle \langle e| = \sum_e \frac{V_e \rho V_e^\dagger}{\mathrm{tr} V_e \rho V_e^\dagger} \otimes \mathrm{tr} V_e \rho V_e^\dagger \, |e\rangle \langle e|$$

This channel has the following interpretation: with probability $p_e := \mathrm{tr} V_e \rho V_e^\dagger = \mathrm{tr} V_e^\dagger V_e \rho =: \mathrm{tr} M_e \rho$ the state of the system after the application of this channel is $\rho_e = \frac{V_e \rho V_e^\dagger}{\mathrm{tr} V_e \rho V_e^\dagger}$. This is similar to having a measurement that outputs the post measured state ρ_e with probability p_e.