Q. Inf. Science II (6.443/8.371/18.436) — Spring 2018

Assignment 1b (the written part)

Posted: Wednesday, Feb 14, 2018
Due: **Friday**, *February 23, 2018 at* ***5pm***

Turn in a paper copy of your solutions in the drop box on the third floor between buildings 8 and 16. If you collaborated with other students, please list their names, and make sure that you write up solutions on your own. The online part of the assignment can be found here.

1. **Cloning and Error Correction**

   (a) Given a set of orthogonal quantum states $|\alpha_1\rangle, \ldots, |\alpha_k\rangle \in \mathbb{C}^d$, where $1 \leq k \leq d$, find the Kraus operators of a quantum channel $\mathcal{N} : L(\mathbb{C}^d) \to L(\mathbb{C}^d \otimes \mathbb{C}^d)$ such that $\mathcal{N}(|\alpha_i\rangle\langle\alpha_i|) = |\alpha_i\rangle\langle\alpha_i| \otimes |\alpha_i\rangle\langle\alpha_i|$ for all $i \in [k]$.

   (b) Show that there is no quantum channel $\mathcal{N} : L(\mathbb{C}^d) \to L(\mathbb{C}^d \otimes \mathbb{C}^d)$ mapping $|\psi\rangle\langle\psi| \to |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$ for all states $|\psi\rangle$. This is called the *no-cloning theorem.* (Hint: use the Stinespring representation of $\mathcal{N}$. Also you may find it helpful to first use the Schmidt decomposition to prove the following fact about the partial trace: if $\mathrm{tr}_B[|\Psi\rangle\langle\Psi|_{AB}] = |\alpha\rangle\langle\alpha|_A$, then $|\Psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B$ for some pure state $|\beta\rangle_B$.)

   (c) Suppose $\{|\bar{0}\rangle, |\bar{1}\rangle\}$ is a quantum code encoding a single logical qubit into n physical qubits that can correct erasures of up to two physical qubits in known locations. (That is, for every pair of locations i, j there is a channel $\mathcal{F}_{ij}$ that recovers the encoded state $|\psi\rangle\langle\psi|$ from the reduced state $\sigma = \mathrm{tr}_{ij}|\psi\rangle\langle\psi|$.) For classical coding the three-bit repetition code can correct two erasure errors. Show that for this quantum coding problem the number of physical qubits n is at least 5.

2. **Computing without Error Correction** In this problem, we study what happens when we don't use quantum error correction. Suppose we wish to implement a unitary U through a sequence of local gates $U_1, U_2, \ldots, U_k$ such that $U = U_k U_{k-1} \ldots U_1$. However, we do not have access to the exact gates $U_1, \ldots, U_k$ but to faulty gates $\tilde{U}_1, \ldots, \tilde{U}_k$ with $\|\tilde{U}_i - U_i\| \leq \epsilon$ for every i. (Here we use $\|\cdot\|$ to denote $\|\cdot\|_\infty$, meaning the largest singular value of a matrix.) Show that

$$\|U - \tilde{U}_k \tilde{U}_{k-1} \ldots \tilde{U}_1\| \leq k\epsilon.$$

   Hint: For unitaries V, W and an arbitrary matrix X, how are $\|X\|$ and $\|VXW\|$ related?