6.251/15.081J Recitation 6

David B. Brown*

October 10, 2002

1 Review Problems

Example 1.1. For each of the following claims, state whether true or false and justify briefly. Assume standard form unless otherwise stated.

(a) $\mathcal{N}(A)$ is a subspace.

(b) A rank-deficient matrix always has a nullspace of dimension greater than zero.

(c) The system $A\boldsymbol{x} = \mathbf{0}$ has a nonzero solution if and only if the rows of A are linearly dependent.

(d) A nonempty polyhedron in standard form can contain at most $\binom{n}{m}$ vertices, and there are LP instances for which this bound is tight.

(e) Degenerate BFSs result in a standard form system if and only if the system has redundant constraints.

(f) A BFS for a standard form system is degenerate if and only if multiple bases correspond to it.

(g) The polyhedron $\{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \mathbf{0}\}$ has either zero or one extreme point.

(h) The tableau columns corresponding to slack variables are precisely the columns of B^{-1}.

(i) There exist multiple optimal solutions if and only if there exists a basis with some zero nonbasic reduced costs.

(j) Let $(\boldsymbol{x}_B, \boldsymbol{x}_N)$ be a BFS and $\bar{\boldsymbol{c}}$ be its reduced costs. For any other feasible solution $\boldsymbol{y}$, the cost difference between $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\bar{\boldsymbol{c}}_N^T \boldsymbol{y}_N$.

*dbbrown@mit.edu

(k) Regardless of the pivoting rule employed, the simplex method will never cycle for a problem with one equality constraint.

(l) An algorithm for finding a solution to a set of linear inequalities implies an algorithm for solving an LP, and vice versa.

(m) The system $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$ is feasible if and only if the rows of $\boldsymbol{A}$ are linearly independent.

(n) (Modified from [3], ch. 4). A system of linear inequalities is infeasible if and only if there exists a nonnegative linear combination of the inequalities which is inconsistent by itself.

(o) Dual degeneracy implies primal non-uniqueness.

(p) If the dual has multiple optimal solutions, then all optimal primal BFSs are degenerate.

Example 1.2. (Modified from [3], ch. 2).

Consider a standard form tableau at a non-optimal BFS. Suppose that for the problem we have the constraint $\mathbf{1}^T\boldsymbol{x} \le \alpha$. Let $\bar{c}_s = \min_{1=1,\dots,n}(\bar{c}_i)$ and z be the objective value of the current BFS. Prove that $z + \alpha\bar{c}_s$ is a lower bound to the optimal value of the problem.

Example 1.3. (From [4], ch. 3).

Suppose an LP in standard form has two columns that are proportional with a positive constant of proportionality. Construct an equivalent LP with one fewer column.

Example 1.4. [1], exercise 4.19.

Example 1.5. (From [2], ch. 4).

Interpretation of LP dual via relaxed problems. Consider the inequality form LP

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \end{array}$$

with optimal value $p^\star$. Suppose that $\boldsymbol{w} \in \mathbb{R}^m$ satisfies $\boldsymbol{w} \ge \mathbf{0}$. If $\boldsymbol{x}$ is feasible for the LP, it also satisfies the inequality

$$\boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{w}^T\boldsymbol{b}.$$

Geometrically, this means that for any $\boldsymbol{w} \ge \mathbf{0}$, the halfspace

$$H_w = \{\ \boldsymbol{x} \in \mathbb{R}^n \mid (\boldsymbol{A}^T\boldsymbol{w})^T\boldsymbol{x} \le \boldsymbol{w}^T\boldsymbol{b}\}$$

contains the feasible set for the LP. Therefore if we minimize the objective $\boldsymbol{c}^T\boldsymbol{x}$ over the halfspace H_w we get a lower bound on $p^\star$.

(a) Derive an expression for the minimum value of $\boldsymbol{c}^T\boldsymbol{x}$ over the halfspace H_w as a function of $\boldsymbol{w} \geq \mathbf{0}$.

(b) Now formulate the problem of finding the best such bound, by maximizing the bound obtained by choice of $\boldsymbol{w} \geq \mathbf{0}$.

(c) Relate the result of **(b)** to the dual of the LP.

References

[1] Bertsimas, D.; Tsitsiklis, J.N. *Introduction to Linear Optimization.* Athena Scientific, 1997.

[2] Boyd, S., Vandenberghe, L. *Convex Optimization.* Course reader. Stanford University, 2001.

[3] Murty, K.G. *Linear and Combinatorial Programming.* John Wiley & Sons, 1976.

[4] Papadimitriou, C.H.; Steiglitz, K. *Combinatorial Optimization: Algorithms and Complexity.* Prentice-Hall, 1982.