6.251/15.081J Quiz 3 Solutions

Massachusetts Institute of Technology

December 11, 2002

Problem 1.

(a) Person 2 can be assigned project 1 with profit $p_1 - c_{21} = 3 - 2 = 1$, or project 3 with profit $3.25 - 2 = 1.25$. Person 4 can be assigned project 3 with profit $3.25 - 2 = 1.25$, or project 4 with profit $3.5 - 3 = 0.5$. Thus, person 4 must be assigned project 3 in order to satisfy 1/4-complementary slackness. This leaves person 2 with project 1, which is fine, since $1 \geq 1.25 - 0.25$.

(b) Persons 1 and 3 make bids. Person 1 will bid on project 1, since $3 - 1 > 2.5 - 1$. 1's bid will be of value $3 - (2 - 1.5) - 0.25 = 2.25$. Person 3 will bid on project 4, since $3.5 - 2 > 2.5 - 2$. 3's bid will be of value $3.5 - (1.5 - .5) - 0.25 = 2.25$ also. Since there are no conflicting bids, the assignments will be updated to person 1 to project 1, person 2 unassigned, person 3 to project 4, and person 4 remains on project 3. The prices are now updated to $p_1 = 2.25$, $p_2 = 2.5$, $p_3 = 3.25$, and $p_4 = 2.25$.

Problem 2.

We define three binary variables for each node i, x_i^1, x_i^2, and x_i^3. Variable x_i^j is 1 if node i is assigned label $j \in \{1, 2, 3\}$ and 0 otherwise[1]. Let E denote

[1]Note: A common, incorrect formulation is to define variables $y_i \in \{0, 1, 2\}$ representing which label any node i has, then introduce the constraints $|y_i - y_j| \geq 1 \ \forall \ \{i, j\} \in E$. These constraints, however, do not form a polyhedron (the feasible set is nonconvex). A good rule of thumb to remember is that absolute values in constraints should always be on the "less than" side of an inequality; to see this, one need only examine the regions defined by $|x| \geq 1$ versus $|x| \leq 1$.

the set of edges in the graph. The problem now has the following form:

$$\begin{array}{rc} \text{minimize} & 0 \\ \text{subject to} & x_i^1 + x_i^2 + x_i^3 = 1, \quad i = 1, \ldots, n, \\ & x_k^j + x_l^j \leq 1, \quad j = 1, 2, 3, \ \forall \{k, l\} \in E, \\ & x_i^j \in \{0, 1\} \quad j = 1, 2, 3, \ i = 1, \ldots, n. \end{array}$$

Problem 3.
Consider a nonbasic arc (i, j), (i.e., outside the tree). It suffices to show that the reduced cost of any such arc is nonnegative. Consider the path from i to j along the tree. Let F and B be the forward and backward arcs on that path, respectively. We have $c_{kl} \leq p_k - p_l + \epsilon, \forall (k, l) \in F$, and $c_{kl} \geq p_k - p_l, \forall (k, l) \in B$ (dual feasibility). Adding the arc costs along the path, we have $\sum_{(k,l) \in F} c_{kl} - \sum_{(k,l) \in B} c_{kl} \leq p_i - p_j + (n-1)\epsilon \leq c_{ij} + (n-1)\epsilon$, where the last inequality uses dual feasibility. Since the arc costs are all integer, and $(n-1)\epsilon < 1$, we obtain $\sum_{(k,l) \in F} c_{kl} - \sum_{(k,l) \in B} c_{kl} \leq c_{ij}$. This shows that the reduced cost of arc (i, j) is nonnegative. Since this holds for every arc outside the tree, we conclude that the corresponding basic feasible flow vector is optimal.

Problem 4.
(a) We need to show that $\boldsymbol{x} \in S_{IP} \Rightarrow \boldsymbol{x} \in S(\boldsymbol{p})$. We have

$$\begin{aligned} \lfloor \boldsymbol{p}^T A \rfloor \boldsymbol{x} &\leq \boldsymbol{p}^T A \boldsymbol{x} \quad (\boldsymbol{x} \geq \boldsymbol{0}) \\ &\leq \boldsymbol{p}^T \boldsymbol{b} \quad (A\boldsymbol{x} \leq \boldsymbol{b}, \ \boldsymbol{p} \geq \boldsymbol{0}) \end{aligned}$$

So we have $\lfloor \boldsymbol{p}^T A \rfloor \boldsymbol{x} \leq \boldsymbol{p}^T \boldsymbol{b}$. On the other hand, $\boldsymbol{x}$ is integer, and so is $\lfloor \boldsymbol{p}^T A \rfloor$, so we have

$$\lfloor \boldsymbol{p}^T A \rfloor \boldsymbol{x} \leq \lfloor \boldsymbol{p}^T \boldsymbol{b} \rfloor,$$

which implies that $\boldsymbol{x} \in S(\boldsymbol{p})$.

(b) The result as stated in the problem is actually false, so this problem was not counted in the grading.[2] If, on the other hand, the problem had

[2] To see that it is not true, consider $A = [2 \ 1]$ and $b = 1$. Then $CH(X) = \{\boldsymbol{x} \mid 2x_1 + x_2 \leq 1\}$, and $CH(X) \cap \{\boldsymbol{x} \mid \boldsymbol{x} \geq \boldsymbol{0}\} = CH(\{(0,0), (1/2, 0), (1, 0)\})$. On the other hand, with $p = 1/2$ in the other feasible set, we get the constraint $x_1 \leq 0$, so we immediately see this feasible set is smaller.

been stated "$\forall\, \boldsymbol{p}$ such that $\boldsymbol{p} \geq \mathbf{0}$ *and* $\boldsymbol{p}^T A$ integral," then the claim is true, as we will now show.

Recall that we can write z_D as the optimal value of the problem:

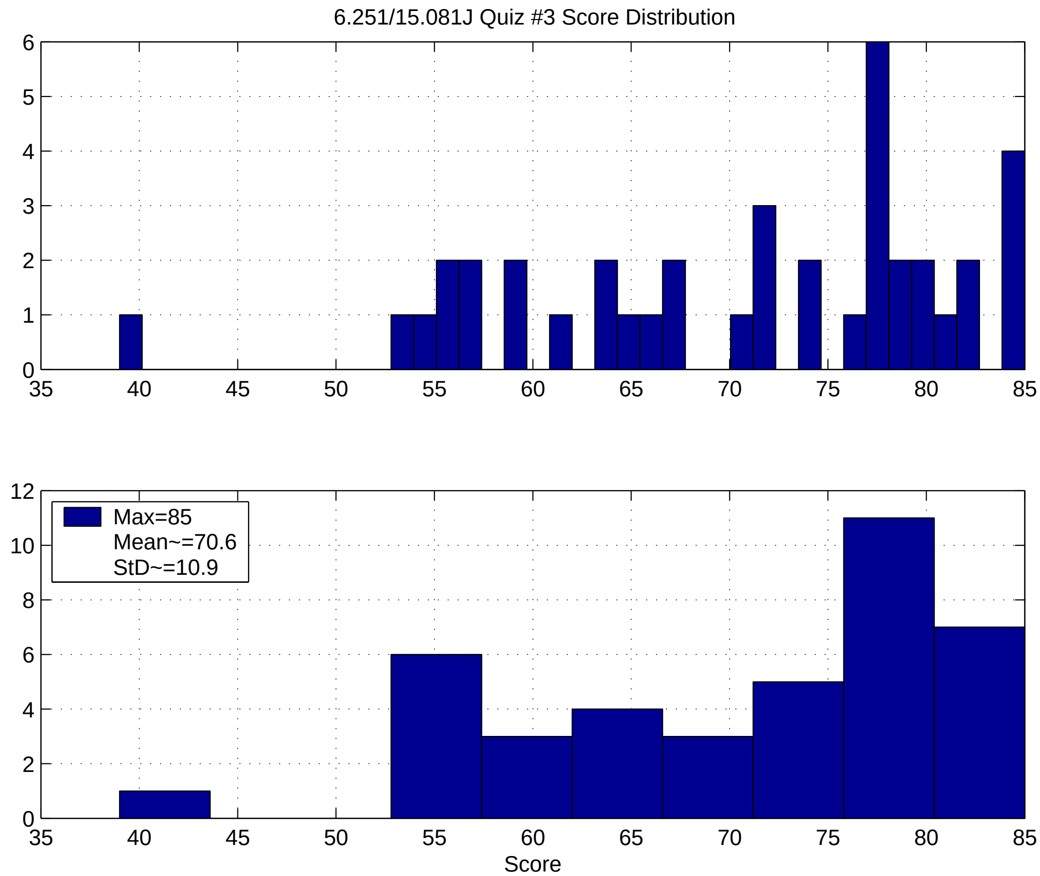
$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{x} \geq \mathbf{0}, \\ & \boldsymbol{x} \in CH\,(X)\,, \end{aligned}$$

where $X = \{\boldsymbol{x} \text{ integer} \mid A\boldsymbol{x} \leq \boldsymbol{b}\}$. Consider the optimal solution $\boldsymbol{x}^\star$ corresponding to z_D. Since $\boldsymbol{x}^\star \in CH\,(X)$, it can be written as a convex combination of the set J of extreme points $\boldsymbol{x}^j$ of $CH(X)$ (we need not consider extreme rays since X is a bounded set by the assumption). Since the $\boldsymbol{x}^j$ are integral (by definition of X), we have that

$$\boldsymbol{p}^T A \boldsymbol{x}^j \leq \lfloor \boldsymbol{p}^T \boldsymbol{b} \rfloor, \quad \forall j \in J.$$

This inequality is valid since $\boldsymbol{x}^j$ satisfies $A\boldsymbol{x}^j \leq \boldsymbol{b}$, $\boldsymbol{p} \geq \mathbf{0}$, and $\boldsymbol{p}^T A$ and $\boldsymbol{x}^j$ are integral, so we can round down the right-hand side with impunity. It clear that the convex combination $\boldsymbol{x}^\star = \sum_{j \in J} \lambda_j \boldsymbol{x}^j$ will satisfy $\boldsymbol{p}^T A \boldsymbol{x}^\star \leq \lfloor \boldsymbol{p}^T \boldsymbol{b} \rfloor$, since the multipliers λ_j are nonnegative and sum to one. This implies that $\boldsymbol{x}^\star$ is feasible to the second optimization problem, so $z_P \leq z_D$.[3]

[3]Even though the claim is false, a disturbing number of people argued along the lines that $CH(X) \cap \{\boldsymbol{x} \mid \boldsymbol{x} \geq \mathbf{0}\} = S_{IP}$. This is **not** true, as can be seen again by the example $A = [2\ 1]\,,\ b = 1$.


6.251/15.081J Quiz #3 Score Distribution
Max=85
Mean~=70.6
StD~=10.9
Score