MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2002 6.251/15.081

Quiz 3 12/11/02 (2:35-3:55 p.m., in class)

**Problem 1.** (25 points)
Consider the assignment problem with arc costs $c_{ij}$ as indicated in Figure 1 and suppose that the projects' prices (rewards) are also as shown.

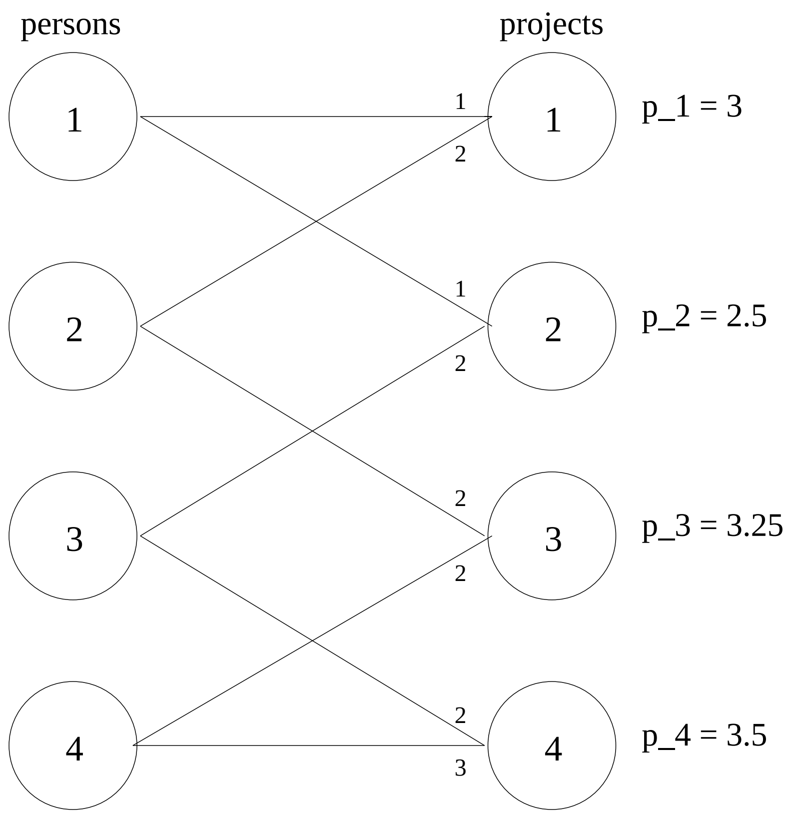


Figure 1: Assignment scenario for problem 1.

(a) Construct a partial assignment involving only persons 2 and 4 that satisfies $\epsilon$-complementary slackness, for $\epsilon = 1/4$.

(b) Starting from these prices and this partial assignment, and with $\epsilon = 1/4$, carry out one iteration of the auction algorithm.

**Problem 2.** (25 points)
We are given an undirected graph with $n$ nodes, and three possible labels. We are interested in deciding whether we can associate a label to each node, so that any two nodes connected by an edge have different labels. Formulate an integer linear programming problem which is feasible if and only if such a labeling is possible.

**Problem 3.** (25 points)
Consider an uncapacitated network flow problem with integer data (i.e., the supplies $b_i$ and the arc costs $c_{ij}$ are all integer). Suppose that we have a nondegenerate basic feasible solution $\mathbf{f}$ and a dual feasible solution $\mathbf{p}$ that satisfies satisfy $\epsilon$-complementary slackness:

$$\text{if } f_{ij} > 0, \qquad \text{then } p_i \geq c_{ij} + p_j - \epsilon.$$

Assume that $\epsilon < 1/(n-1)$. Show that $\mathbf{f}$ is optimal. [*Hint:* Say something about the cost of interesting cycles; alternatively, bound the cost of $\mathbf{f}$.]

**Problem 4.** (25 points)
Let

$$S_{\text{IP}} = \{\mathbf{x} \text{ integer} \mid \mathbf{A}\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}.$$

Also, given a vector $\mathbf{p}$, let

$$S(\mathbf{p}) = \{\mathbf{x} \mid \lfloor \mathbf{p}'\mathbf{A} \rfloor \mathbf{x} \le \lfloor \mathbf{p}'\mathbf{b} \rfloor\}.$$

(a) Show that if $\mathbf{p} \ge \mathbf{0}$, then

$$S_{\text{IP}} \subseteq S(\mathbf{p}).$$

(b) Assume that the set $\{\mathbf{A}\mathbf{x} \le \mathbf{b}, \mathbf{x} \text{ integer}\}$ is finite. Let

$$z_D = \max_{\mathbf{q} \ge \mathbf{0}} \; \min\{\mathbf{c}'\mathbf{x} - \mathbf{q}'\mathbf{x} \mid \mathbf{A}\mathbf{x} \le \mathbf{b}, \mathbf{x} \text{ integer}\},$$

be the value of the Lagrangean dual, if we relax and dualize the constraints $\mathbf{x} \ge \mathbf{0}$. Also, let

$$\begin{aligned} z_P = \quad & \min && \mathbf{c}'\mathbf{x} \\ & \text{subject to} && \lfloor \mathbf{p}'\mathbf{A} \rfloor \mathbf{x} \le \lfloor \mathbf{p}'\mathbf{b} \rfloor, \quad \forall \mathbf{p} \ge \mathbf{0} \\ & && \mathbf{x} \ge \mathbf{0}. \end{aligned}$$

Show that $z_P \le z_D$.

*Note:* The optimization problem that defines $z_P$ involves infinitely many constraints. It turns out that these constraints define a polyhedron, but the problem can be solved without appealing to this fact.