6.251/15.081J Quiz 2 Solutions

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Problem 1.

(a) There exists a maximum flow of value $+\infty$ if and only if there exists an $s-t$ augmenting path P with $\delta(P) = \infty$, where s and t are the source and destination nodes of the graph, respectively. Consider a variant of the labeling algorithm whereby, instead of accepting unsaturated arcs when we scan each node, we only accept forward arcs with infinite capacity. It is clear that this modified algorithm will find an $s-t$ path of infinite capacity if and only if such a path exists.

(b) Since the B matrix in an uncapacitated network flow problem is the incidence matrix (less the constraint from node n) of a (directed) tree, we know that we can always permute the rows of B to make a lower triangular matrix with diagonal entries equal to 1 or -1. It is then clear from Cramer's rule that such a matrix has determinant 1 or -1, and therefore its inverse will be integral.

(c) An extreme ray for any standard form problem will be an extreme ray of the recession cone $C = \left\{\boldsymbol{x} \in \mathbb{R}^n_+ \mid \mathsf{A}\boldsymbol{x} = \mathbf{0}\right\}$. In the case of network flow problems, A is the incidence matrix, so we can picture the extreme rays of C in this case to be cycles in the graph ($\mathsf{A}\boldsymbol{x} = \mathbf{0}$) such that all the arcs point in the same direction ($\boldsymbol{x} \geq \mathbf{0}$).

Problem 2.

We construct a maximum flow graph G in the following way. We introduce nodes $c_1, \ldots, c_n$ for each contractor, and connect a directed arc of unit capacity from s to each of these nodes. We also introduce nodes $p_1, \ldots, p_n$ for each project, and connect a directed arc of unit capacity from each p_i to t.

Finally, we introduce an infinite capacity arc from each contractor node c_i to each project node in $A(i)$. Let C be the set of contractor nodes, and P be the set of project nodes, respectively. It is clear that there exists a matching in which each project gets assigned if and only if the maximum flow value is n.

Consider a minimum cut R in G, with $s \in R$, $t \notin R$. Denote the set of project nodes not in R by S, i.e., $S = P \cap (R \setminus P)$. We claim that the set V of contractor nodes not in R is given by $\{i \in C \mid A(i) \cap S \neq \emptyset\}$. First, it is clear that any node in $\{i \in C \mid A(i) \cap S \neq \emptyset\}$ must be included in V, since otherwise the cut would cross an outward infinite capacity arc. Conversely, assume we have a node j not in $\{i \in C \mid A(i) \cap S \neq \emptyset\}$ but in V. By adding j to R, we decrease the capacity of the cut by 1, which contradicts that R is minimum. So we have proved that the set V of contractor nodes not in R is given by $V = \{i \in C \mid A(i) \cap S \neq \emptyset\}$.

Now, the capacity of this cut is given by $\delta(R) = |V| + n - |S|$. If there does not exist a matching in which each project is assigned, then we know the maximum flow value, and hence, the minimum cut capacity, is strictly less than n. This implies that $|V| < |S|$, and from the derivation of V from the preceding paragraph, we have that S is undersubscribed. Conversely, if there exists a set S which is undersubscribed, then the cut derived by S and the corresponding V satisfy $\delta(R) < n$, which implies that the maximum flow value is also less than n, so there does not exist a matching in which each project is assigned.

Problem 3.

(a) $\lambda_1^1 = \lambda_1^2 = 1/2$ and $\lambda_2^1 = 1$ satisfy the flow conservation constraints as well as the coupling constraint on f_{23}.

(b) We have three basic variables, so there must be three constraints in the basis matrix. In particular, we have the single coupling constraint and the two convexity constraints (one for each subproblem). Plugging in the values of the extreme points into the coupling constraint, we have the basis B as

$$\mathsf{B} = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) We have $[\boldsymbol{q}\ r_1\ r_2] = \boldsymbol{p}^T$, where $\boldsymbol{p} = \left(B^{-1}\right)^T \boldsymbol{c}_B$ is the vector of dual variables from the problem. Plugging in the given extreme points and noting the cost coefficients for each of the graphs, we have that $c_1^1 = 20$, $c_1^2 = 12$, and $c_2^1 = 0$. Now solving the system for $\boldsymbol{p}$, we find $q = 2$, $r_1 = 12$, and $r_2 = 4$.

(d) We write the second subproblem as

$$\begin{aligned} \text{minimize} \quad & \left(\boldsymbol{c_2}^T - qD_2\right) \boldsymbol{f} \\ \text{subject to} \quad & A\boldsymbol{f} = \boldsymbol{b} \\ & \boldsymbol{f} \geq \boldsymbol{0}, \end{aligned}$$

where A is the truncated incidence matrix of network 2, $\boldsymbol{b}$ are the supplies of network 2, and $D_2 = [0\ \ 0\ 0\ -1]$ (from the coupling constraint). In terms of numbers, the second subproblem is

$$\begin{aligned} \text{minimize} \quad & \begin{bmatrix} 1 & 0 & 0 & 2 \end{bmatrix} \boldsymbol{f} \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \boldsymbol{f} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ & \boldsymbol{f} \geq \boldsymbol{0}. \end{aligned}$$

It is not difficult to solve this by inspection. We look for an optimal basis. Clearly, columns 2 and 3 are linearly dependent, so they cannot form a basis. Also, column 4 has a high cost coefficient, so we consider $\{1, 2\}$ or $\{1, 3\}$ for the basis. The former is infeasible, but $\{1, 3\}$ is feasible and in fact optimal, with $\boldsymbol{f} = [f_{13}\ f_{12}\ f_{21}\ f_{23}]^T = [2\ 0\ 1\ 0]$. Since the cost of this solution is less than $r_2 = 4$, we bring this new extreme point $\boldsymbol{f}_2^2$ into the master problem.

It is not hard to see that the next solution to the master problem will have $\lambda_1^1 = 0$, $\lambda_1^2 = 1$, $\lambda_2^1 = 0$, and $\lambda_2^2 = 1$ (so it is degenerate). The new arc flows now read

$$\begin{bmatrix} f_{13} \\ f_{12} \\ f_{23} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

for network 1, and

$$\begin{bmatrix} f_{13} \\ f_{12} \\ f_{21} \\ f_{23} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

for network 2.