6.251/15.081: Supplement to the 9/11/02 lecture

The theorem here is the one discussed and partially proved in lecture. It combines Theorems 2.6 and 2.8 in the text, and of course subsumes Theorem 2.7, which is a special case of Theorem 2.8. (Still, the proof of Theorem 2.7 is worth reading because it brings in a different idea.)

Theorem: Consider a polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$, and assume it is nonempty. Then, the following are equivalent.
(i) P does not contain a line.
(ii) For any cost vector $\mathbf{c} \in \Re^n$, either the optimal cost is $-\infty$, or there exists an extreme point which is optimal.
(iii) There exist n vectors out of the family $\mathbf{a}_1, \ldots, \mathbf{a}_m$, which are linearly independent.

There are three implications to be proved. We start with the easier two.

(ii)⇒ (iii) We assume that P has property (ii) in the theorem. This property asserts something for every $\mathbf{c}$. As a special case, when $\mathbf{c} = \mathbf{0}$, it asserts that either the optimal cost is $-\infty$, or the optimum attained at some extreme point. But when $\mathbf{c} = \mathbf{0}$, the optimal cost is zero (as is the cost of any feasible solution), and so the optimum is attained at an extreme point. In particular, there must exist at least one extreme point. We know that an extreme point is the same as a basic feasible solution. Recall that at a basic feasible solution, we can find n of the constraints that are active and for which the corresponding vectors are linearly independent. This implies that the family $\mathbf{a}_1, \ldots, \mathbf{a}_m$ does contain n linearly independent vectors.

(iii)⇒ (i) This is the same as the implication (c)⇒(b) in Theorem 2.6, proved in p. 64 of the text.

(i)⇒ (ii) The statement to be proved here is exactly the same as Theorem 2.8, except that the assumption "Suppose that P has at least one extreme point" is now replaced by the assumption that P has no lines. But if you read the proof of Theorem 2.8 in the text carefully, you will see that the assumption that P has an extreme point is used in only one place: in the statement "Since P has no lines..." in p. 67, 2nd line, which is an assertion made possible by the earlier Theorem 2.6. Here, however, we have already assumed that P has no lines, and so the same proof goes through, without the help from Theorem 2.6.