DAVE'S SIMPLIFIED MODEL OF HEAT LOSS FROM A TO-GO COFFEE CUP

**Summary**
To provide a simple model of the cooling of coffee, the one-dimensional form of Fourier's Law of thermal conduction is applied to a simple model of a coffee cup. Assumptions and simplifications are outlined; variables and constants are identified; and the differential equation that results from Fourier's Law is solved, yielding the temperature as a function of time. This solution can be used to determine how much cardboard insulation is needed to keep coffee hot for a given time and temperature.

**Model**
A typical to-go cup of coffee looks something like the image shown in Figure 1. The coffee contains thermal energy, $Q$, in proportion to its mass, $m$, its heat capacity, $c_h$, and its temperature, $T$ (Eq. 1).

$$Q = mc_hT \tag{1}$$

Presuming that 1) the energy transferred to the cup of coffee is negligible (no fire underneath the coffee, no radio-active coffee grounds imparting high-velocity protons to coffee, no parabolic mirror focusing sunshine on the coffee…) and 2) the temperature of the coffee is higher than the surrounding environment, thermal energy transfers from the coffee to the environment around the coffee, which is presumed to be arbitrarily large such that as the coffee cools the temperature of the environment does not change noticeably. The energy transfer predominantly occurs through three processes, radiation, convection, and conduction.


**Figure 1** A typical coffee cup and the primary paths of heat loss

*Radiation*

Check for yourself that radiation will not significantly contribute to the heat loss in coffee as per Eq. 2.

$$\frac{dQ}{dt} = -As\left(T_{coffee}^4 - T_{room}^4\right) \tag{2}$$

*s*, the Stefan-Boltzmann constant, $5.7 \times 10^{-8}$ W/m$^2$•°K$^4$, *A*, the area of the radiating surface

Bottom line: for keeping coffee hot by insulation, you can ignore radiative heat loss.

*Convection*

Two sorts of convection are conveniently ignored by this simplification as shown in Figure 1.

- The first is that of mass transport, whereby hot coffee molecules (steam) escape the coffee cup and mix with the air. Mass transport is not considered by this model because it is assumed that there will be a lid on the coffee cup. And I think this heat loss has been characterized during the explorations.
- The second form of convection occurs at the boundary between the cardboard and the air and is due to the heating and movement of air molecules near the surface. The convective conductance is *hA*, where *h* is the convection coefficient (~10 W/m$^2$•°C) and *A* is the area at the air/cardboard boundary. I'll argue here that this conductance is pretty high—effectively what would happen if a coffee cup were made of metal—and thus will contribute little to keeping coffee hot for two hours. For a more complete model, the convection contribution can be included in the conductance analysis as if it were an additional resistive layer. Bottom line: for keeping coffee hot by insulation, you can ignore convection.

*Conduction*

In a simplified model, an insulated coffee cup can be considered as a hot body separated from a cooler environment by an insulating material of thickness *d* and cross-sectional area *A* as shown in Figure 2. This idealization is useful because the one-dimensional form of Fourier's law of cooling can be used to describe the cooling of the coffee, as shown in Equation 3.

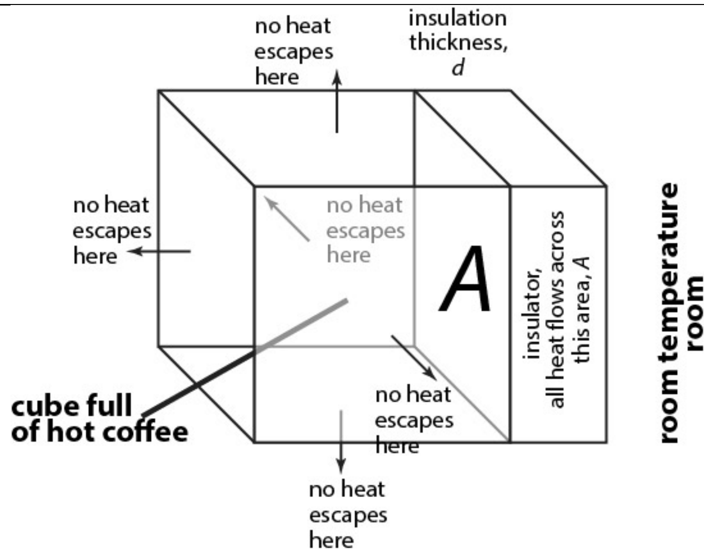


**Figure 2** A idealized coffee cup; heat is only lost through conduction in one direction across an insulator of cross-sectional area, *A*, and thickness *d*.

$$\frac{dQ}{dt} = -\frac{kA}{d}\left(T_{coffee} - T_{room}\right) \quad (3)$$

*k*, the conductivity of the insulator in W/m•°K, other variables as previously described

Substitution of Equation 1 into Equation 3 coupled with the observation that the heat capacity and the mass of the coffee remain constant produces equation 4.

$$mc_h\frac{dT_{coffee}}{dt} = -\frac{kA}{d}\left(T_{coffee} - T_{room}\right) \quad (4)$$

$$mc_h\frac{dU_{(t)}}{dt} = -\frac{kA}{d}\left(U_{(t)}\right) \quad (5)$$

$$\frac{dU_{(t)}}{dt} = -\frac{kA}{dmc_h}\left(U_{(t)}\right) \quad (6)$$

Equation 6 defines a function time, $U_{(t)}$, that is proportional to the negative of its time derivative. Solutions for $U_{(t)}$ may be of the form of $Be^{-t/\tau} + C$, where *B, C* and $\tau$ are constants; $\tau$ is constrained by the geometry and properties of the coffee and its insulator;

and *B* & *C* are determined by the initial conditions (temperatures) of the coffee cup and the room. Algebra and substitution of *T* for *U* yield Equation 7, which predicts the temperature of a cup of coffee as it cools.

$$T_{coffee\ (t)} = T_{room} + \left(T_{coffee\ (at\ time\ 0)} - T_{room}\right)e^{-t/\tau} \qquad (7)$$

$$\text{Where } \tau = \frac{dmc_h}{kA}$$


http://prezi.com/iwto6j4zvzni/heat-transfer-principles-of-coffee-cups/
http://www.engineeringtoolbox.com/convective-heat-transfer-d_430.html
http://www.efunda.com/formulae/heat_transfer/home/glossary.cfm?ref=stefan#stefan
http://en.wikipedia.org/wiki/Thermal_conduction