18.06 Professor Strang Exam 3 Solutions May 6, 1998

1. (a)

$$\begin{aligned} \lambda_1 &= 1 & \lambda_2 &= 0 \\ x_1 &= a = \begin{bmatrix} 3 \\ 4 \end{bmatrix} & x_2 &= \begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{aligned}$$

(or multiples of x_1 and x_2)

(b)

$$\begin{aligned} \lambda_1 &= .6 + .8i & \lambda_2 &= .6 - .8i \\ x_1 &= \begin{bmatrix} 1 \\ -i \end{bmatrix} & x_2 &= \begin{bmatrix} 1 \\ i \end{bmatrix} \end{aligned}$$

(c)

$$\begin{aligned} \lambda_1 &= 2(1) - 1 = 1 \\ \lambda_2 &= 2(0) - 1 = -1 \end{aligned}$$

R has the same eigenvectors as P

2. (a)

$$\begin{aligned} \lambda_1 &= 1 && \text{(for any Markov matrix)} \\ \lambda_2 &= 0 && \text{(since } A \text{ is singular)} \\ \lambda_3 &= .6 && \text{(since the trace of } A \text{ is 1.6)} \end{aligned}$$

(1, 2, 3 can be permuted)

(b)

$$\begin{aligned} n \geq 1: \quad A^n u_0 &= 1^n x_1 + 0^n x_2 + (.6)^n x_3 \\ &= x_1 + (.6)^n x_3 \end{aligned}$$

(c)

$A^n u_0$ approaches x_1 .

3. (a) Suppose M is any invertible matrix. Circle all the properties of a matrix A that remain the same for $M^{-1}AM$:

   $\boxed{\text{same rank}}$

   same nullspace

   $\boxed{\text{same determinant}}$

   $\boxed{\text{real eigenvalues}}$

   orthonormal eigenvectors

   symmetric positive definiteness

   (b) This is a similar question but now Q is an orthonormal matrix. Circle the properties of A that remain the same for $Q^{-1}AQ$:

$$= Q^T AQ\ !$$

   same column space

   $\boxed{A^k \text{ approaches zero as } k \text{ increases}}$

   $\boxed{\text{orthonormal eigenvectors}}$

   $\boxed{\text{symmetric positive definiteness}}$

   $\boxed{\text{projection matrix}}$

4. (a) Suppose the 5 by 4 matrix A has independent columns. What is the most information you can give about

   the eigenvalues of A^TA: **They are real and positive**

   the eigenvectors of A^TA: **They are orthogonal**

   the determinant of A^TA: **The determinant is positive**

   (b)

$$A = \underset{U}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

   (c)

$$\begin{aligned} T(v_1 + v_2) &= T(v_1) + T(v_2) \\ &= Iw_1 + Iw_2 = w_1 + w_2 \\ &\quad \text{(intermediate steps may be ommitted)} \end{aligned}$$