Final Examination in Linear Algebra: 18.06

May 18, 1998 Solutions Professor Strang

1. (a) zero vector $\{0\}$

   (b) $5 - 4 = 1$

   (c) $x_p = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

   (d) $x = x_p$ because $\boldsymbol{N}(A) = \{0\}$.

   (e) $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

2. (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   The free variable is x_2. The complete solution is

$$x = x_p + x_n = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

   (b) A basis is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

3. (a) The columns of B are a basis for the row space of A (because the row space is the orthogonal complement of the nullspace).

   (b) N is n by $(n - r)$; B is n by r.

4. (a) $\det A = 6$

   (b) $\det B = 6$

   (c) $\det C = 0$

5. (a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

All four matrices are only examples (many other correct answers exist).

6. (a) $\text{rank}(B) = 1$; all multiples of u_1 are in the column space; the vectors $v_1 - v_2$ and v_3 are a basis for the nullspace.

(b) $AA^T = (u_1 v_1^T + u_2 v_2^T)(v_1 u_1^T + v_2 u_2^T) = u_1 u_1^T + u_2 u_2^T$ since $v_1^T v_2 = 0$. AA^T is symmetric and it equals $(AA^T)^2$:

$$(u_1 u_1^T + u_2 u_2^T)(u_1 u_1^T + u_2 u_2^T) = u_1 u_1^T + u_2 u_2^T$$

(The eigenvalues of AA^T are $1, 1, 0$)

(c) $A^T A = v_1 v_1^T + v_2 v_2^T$ since $u_1^T u_2 = 0$.

$$\begin{aligned} A^T A v_1 &= (v_1 v_1^T + v_2 v_2^T) v_1 = v_1 \\ A^T A v_2 &= (v_1 v_1^T + v_2 v_2^T) v_2 = v_2 \,. \end{aligned}$$

Since v_1, v_2 are a basis for $\mathbf{R}^2$, $A^T A v = v$ for all v.

7. (a) $Ax = b$ is $\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ B \end{bmatrix}$. This is solvable if $\underline{B = 2}$.

(b) $A^T A\bar{x} = A^T b$ is

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \overline{C} \\ \overline{D} \end{bmatrix} = \begin{bmatrix} 1+B \\ B \end{bmatrix}.$$

Then $\overline{C} = \frac{1+B}{3}$ and $\overline{D} = \frac{B}{2}$

(c) $A^T A\bar{x} = A^T b$ is

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad p = A\bar{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/3 \\ 5/6 \end{bmatrix}.$$

(d) $Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ columns were already orthogonal, now orthonormal

8. (a) $\lambda = 1, 1, 4$. Eigenvectors can be

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(could also be chosen orthonormal because $A = A^T$)

(b) Circle all the properties of this matrix A:

A is a projection matrix

$\boxed{A \text{ is a positive definite matrix}}$

A is a Markov matrix

A has determinant larger than trace

$\boxed{A \text{ has three orthonormal eigenvectors}}$

$\boxed{A \text{ can be factored into } A = LU}$

(c)

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^{100} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 1^{100} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4^{100} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4^{100}+1 \\ 4^{100}-1 \\ 4^{100} \end{bmatrix}.$$