Final Examination in Linear Algebra: 18.06

May 18, 1998   9:00–12:00   Professor Strang

Your name is: ______________________________

Grading 1 2 3 4 5 6 7 8

Please circle your recitation:

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7)	T2	2-132	Y. Ma	2-333	3-7826	yanyuan@math

Answer all 8 questions on these pages. This is a closed book exam. Calculators are not needed in any way and therefore not allowed (to be fair to all). *Grades are known only to your recitation instructor.* Best wishes for the summer and thank you for taking 18.06.

GS

1 If A is a 5 by 4 matrix with linearly independent columns, find each of these **explicitly**:

(a) **(3 points)** The nullspace of A.

(b) **(3 points)** The dimension of the left nullspace $\boldsymbol{N}(A^T)$.

(c) **(3 points)** One particular solution x_p to $Ax_p =$ column 2 of A.

(d) **(3 points)** The general (complete) solution to $Ax =$ column 2 of A.

(e) **(3 points)** The reduced row echelon form R of A.

2 (a) **(5 points)** Find the general (complete) solution to this equation $Ax = b$:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

(b) **(3 points)** Find a basis for the column space of the 3 by 9 block matrix $[A\ 2A\ A^2]$.

3 (a) **(5 points)** The command $N =$ **null** (A) produces a matrix whose columns are a basis for the nullspace of A. What matrix (describe its properties) is then produced by $B =$ **null** (N')?

(b) **(3 points)** What are the shapes (how many rows and columns) of those matrices N and B, if A is m by n of rank r?

4 Find the determinants of these three matrices:

(a) **(2 points)**

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(b) **(2 points)**

$$B = \begin{bmatrix} 0 & -A \\ I & -I \end{bmatrix} \qquad \text{(8 by 8, same } A\text{)}$$

(c) **(2 points)**

$$C = \begin{bmatrix} A & -A \\ I & -I \end{bmatrix} \qquad \text{(8 by 8, same } A\text{)}$$

5 **If possible** construct 3 by 3 matrices A, B, C, D with these properties:

(a) **(3 points)** A is a **symmetric** matrix. Its row space is spanned by the vector $(1, 1, 2)$ and its column space is spanned by the vector $(2, 2, 4)$.

(b) **(3 points)** All three of these equations have **no solution** but $B \neq 0$:

$$Bx = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Bx = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad Bx = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) **(3 points)** C is a real square matrix but its eigenvalues are not all real and not all pure imaginary.

(d) **(3 points)** The vector $(1, 1, 1)$ is in the row space of D but the vector $(1, -1, 0)$ is not in the nullspace.

6 Suppose u_1, u_2, u_3 is an orthonormal basis for $\mathbf{R}^3$ and v_1, v_2 is an orthonormal basis for $\mathbf{R}^2$.

(a) **(5 points)** What is the **rank**, what are **all vectors** in the **column space**, and what is a **basis for the nullspace** for the matrix $B = u_1(v_1 + v_2)^T$?

(b) **(5 points)** Suppose $A = u_1 v_1^T + u_2 v_2^T$. Multiply AA^T and simplify. Show that this is a projection matrix by checking the required properties.

(c) **(4 points)** Multiply $A^T A$ and simplify. This is the identity matrix! Prove this (for example compute $A^T A v_1$ and then finish the reasoning).

7 (a) **(4 points)** If these three points happen to lie on a line $y = C + Dt$, what system $Ax = b$ of three equations in two unknowns would be solvable?

$$y = 0 \text{ at } t = -1\,, \qquad y = 1 \text{ at } t = 0\,, \qquad y = B \text{ at } t = 1\,.$$

Which value of B puts the vector $b = (0, 1, B)$ into the column space of A?

(b) **(4 points)** For every B find the numbers $\overline{C}$ and $\overline{D}$ that give the best straight line $y = \overline{C} + \overline{D}t$ (closest to the three points in the least squares sense).

(c) **(4 points)** Find the projection of $b = (1, 0, 0)$ onto the column space of A.

(d) **(2 points)** If you apply the Gram-Schmidt procedure to this matrix A, what is the resulting matrix Q that has orthonormal columns?

8 (a) **(5 points)** Find a complete set of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(b) **(6 points, 1 each)** Circle all the properties of this matrix A:

A is a projection matrix

A is a positive definite matrix

A is a Markov matrix

A has determinant larger than trace

A has three orthonormal eigenvectors

A can be factored into $A = LU$

(c) **(4 points)** Write the vector $u_0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ as a combination of eigenvectors of A, and compute the vector $u_{100} = A^{100} u_0$.