MIT 18.06 Makeup Exam 3, Spring 2022
Johnson

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/38
2	/32
3	/30
total	/100

Problem 0 (∞ points): Honor code

Copy the following statement with your signature into your solutions:

> *I have completed this exam* ***closed-book/closed-notes*** *entirely on my* ***own.***
>
> [your signature]

Problem 1 (8+7+8+15 points):

A is a **Hermitian** matrix with eigenvectors (each normalized to length $\|x_k\| = 1$) given by the columns of the following matrix (shown to 3 decimal places):

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \approx \begin{pmatrix} 0.236 & 0.247 & 0.676 & 0.154 & 0.634 \\ -0.548 & -0.495 & 0.094 & 0.653 & 0.138 \\ 0.765 & -0.582 & -0.164 & 0.211 & 0.066 \\ 0.117 & -0.078 & 0.655 & 0.100 & -0.736 \\ -0.211 & -0.591 & 0.279 & -0.703 & 0.182 \end{pmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 3$, $\lambda_4 = 2$, and $\lambda_5 = 1$. Using this matrix A, we solve a system of ODEs:

$$\frac{dy}{dt} - \alpha y = Ay$$

for some initial condition $y(0)$ to find $y(t)$ and some **real or complex** number α.

(a) What are the eigenvalues of $X^T X$?

(b) Write the solution as $y(t) = e^{Bt} y(0)$ for some matrix B: give a formula for B in terms of A and α.

(c) Give a value of α that would cause the solution $y(t)$ to **decay to zero** for *all* initial conditions $x(t)$.

(d) For $\alpha = -5$, give a **good approximation** for $y(100)$ if

$$y(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

You can leave your solution in the form of some vector times some coefficient(s) **without carrying out the explicit multiplications**, but give all the numbers in your vector and coefficients to 3 decimal digits.

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Problem 2 (32 points):

A is the matrix

$$A = \begin{pmatrix} -1 & 18 & 4 & 3 & 17 \\ & 3 & 3 & 5 & 1 \\ & & 0 & -1 & 2 \\ & & & 2 & 4 \\ & & & & 1 \end{pmatrix}.$$

(a) What are the eigenvalues of A?

(b) What is $\det((A + 2I)^2)$?

(c) If you solve $\frac{dx}{dt} = -A^T Ax$ for $x(t)$ given some randomly chosen initial condition $x(0)$, would you typically expect the solutions $x(t)$ to **diverge**, **decay to zero**, **approach a nonzero constant vector**, or **oscillate forever** as $t \to \infty$?

(d) If you compute $x_n = (\frac{1}{3}A - \frac{2}{3}I)^n x$ for some randomly chosen initial vector x_0, would you typically x_n to **diverge**, **decay to zero**, **approach a nonzero constant vector**, or **oscillate forever** as $n \to \infty$?

(blank page for your work if you need it)

Problem 3 (30 points):

For each of the following, say what **must** be true of the **eigenvalues** λ of A (which you can assume is **diagonalizable**) if:

(a) $\|e^{(A-I)t}x\| \to \infty$ for **some** x as $t \to \infty$.

(b) $\|e^{(A-I)t}x\| \to \infty$ for **all** $x \neq 0$ as $t \to \infty$.

(c) $\|(I + A^2)^n x\|$ does *not* diverge for **any** x as $n \to \infty$.

(d) A is a Markov matrix but $A^n x$ does **not** approach a constant vector as $n \to \infty$ for some initial x.

(e) A^2 is Hermitian.

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