MIT 18.06 Exam 3 **Solutions**, Spring 2022
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Problem 1 (10+10+10 points):

The matrix

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

has an eigenvalue $\lambda_1 = 1$ and corresponding eigenvector $x_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(a) What is the other eigenvalue λ_2 and a corresponding eigenvector $x_2 = \begin{pmatrix} 1 \\ ?? \end{pmatrix}$?

(b) B is a 2×2 matrix such that $Bx_k = (1-\lambda_k+\lambda_k^2)x_k$ for the two eigenvectors ($k = 1, 2$). What is B?

(c) What is $A^{3/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

Solution:

(a) $\text{trace}(A) = 3+2 = 5 = \lambda_1+\lambda_2$, so the other eigenvalue is $\lambda_2 = 5-\lambda_1 = \boxed{4}$. To find a corresponding eigenvector, we need to solve

$$(A-4I)x_2 = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} x_2 = 0\,.$$

By insspection, the second column is minus the first, so a solution is $\boxed{x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ or any multiple thereof (but you were requested to scale x_2 so that the first component $= 1$).

(b) $Bx_k = (1-\lambda_k+\lambda_k^2)x_k$ is an eigen-equation: B has the same eigenvectors as A but with the eigenvalues replaced by $1-\lambda_k+\lambda_k^2$. That means that

$$B = I - A + A^2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} + \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}}_{\begin{pmatrix} 11 & 5 \\ 10 & 6 \end{pmatrix}} = \boxed{\begin{pmatrix} 9 & 4 \\ 8 & 5 \end{pmatrix}}.$$

You could have also solved this by diagonalization: $B = X \begin{pmatrix} 1-\lambda_2+\lambda_2^2 & \\ & 1-\lambda_2+\lambda_2^2 \end{pmatrix} X^{-1}$ where $X = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$ is the matrix of eigenvectors, but this may be more work since you have to compute X^{-1}, unless you happen to remember the formula for the inverse of a 2×2 matrix.

(c) The key trick, as usual, is that $A^{3/2}$ multiplies an *eigenvector* (where A acts like a scalar) by $\lambda^{3/2}$. So, to apply $A^{3/2}$ to an arbitrary vector, we just expand that vector in the basis of the eigenvectors and then multiply each term by $\lambda^{3/2}$. Here,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{x_1} + c_2 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{x_2} = \underbrace{\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}}_{X} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Proceeding by Gaussian elimination, we add twice the first row to the second row to obtain:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}}_{U} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies c_2 = 1/3,\ c_1 = 1 - 1/3 = 2/3\,.$$

(Yes, the answer requires the dread "fractions." Sorry!) Hence

$$A^{3/2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{2}{3}\lambda_1^{3/2} x_1 + \frac{1}{3}\lambda_2^{3/2} x_2 = \frac{2}{3} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{8}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 10/3 \\ 4/3 \end{pmatrix}}.$$

Problem 2 (7+7+7 points):

A is a square matrix such that $N(A-I)$ is spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $N(A-5I)$ is spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(a) Without much calculation, you can tell that A **is** / **is not** (choose 1) Hermitian because __________________.

(b) What is A? You can leave your answer as a **product of matrices and/or matrix inverses** without multiplying/inverting them.

(c) What is e^{A+I}? You can leave your answer as a **product of matrices and/or matrix inverses** without multiplying/inverting them, but your answer should not have exponentials of matrices or infinite series.

Solution:

(a) The two nullspace vectors are eigenvectors of A with $\lambda = 1$ and 5, respectively, but they are clearly **not orthogonal**, so A is **not** Hermitian.

(b) From the dimensions of the vectors, A must be a 2×2 matrix, and we are given two eigenvectors for two eigenvectors. Hence, it is diagonalizable and

$$A = X\Lambda X^{-1} = \boxed{\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & \\ & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}}.$$

You **weren't required** to simplify it further, but it turns out that $A = \begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}$ if you work it all out.

(c) e^{A+I} has the same eigenvectors as A, with the eigenvalues replaced by $\lambda \to e^{\lambda+1}$. So, we can again use the diagonalization

$$e^{A+I} = \boxed{\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^2 & \\ & e^6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}}.$$

Problem 3 (4+4+4+4+4+4 points):

For each of the following, say whether it **must** be true, it **may** be true, or it **cannot** be true. No justification needed.

(a) If a matrix is diagonalizable, it **must/may/cannot** have orthogonal eigenvectors.

(b) M is a Markov matrix. If $M^n x$ converges to a steady state as $n \to \infty$ for *any* vector x, the M **must/may/cannot** be a positive Markov matrix (i.e. have all entries > 0).

(c) If a matrix A is *not* diagonalizable, then $\det(A - \lambda I)$ **must/may/cannot** have repeated roots.

(d) If $A^n x$ goes to zero as $n \to \infty$ for *some* x, then A **must/may/cannot** have an eigenvalue λ with $|\lambda| > 1$

(e) If $e^{At}x$ goes to zero as $t \to \infty$ for *every* x, then A **must/may/cannot** have an eigenvalue λ with $|\lambda| > 1$

(f) If A has an eigenvector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then it **must/may/cannot** have an eigenvector $\begin{pmatrix} -3 \\ -6 \\ -9 \end{pmatrix}$.

Solution:

(a) **May**. (All "normal" matrices $AA^H = AA^H$, such as Hermitian matrices, are diagonalizable with orthogonal eigenvectors, but the converse is not true: not all diagonalizable matrices are normal. On the other hand, all diagonalizable matrices are *similar* to normal matrices, so there is *some* change of basis in which their eigenvectors are orthogonal.)

(b) **May**. (All *positive* Markov matrices *must* yield a steady state—they have a single $\lambda = 1$ eigenvalue and all others have $|\lambda| < 1$, but the converse is not true: a Markov matrix with zero entries *may* still have a single $|\lambda| = 1$ eigenvalue. On the other hand, although *any* Markov matrix *must* have a $\lambda = 1$ eigenvalue, it *may* also have other eigenvalues like $\lambda = -1$ that can cause $M^n x$ to oscillate forever without converging.)

(c) **Must**. Non-diagonalizable (defective) matrices can only arise when the characteristic polynomial has repeated roots. (The converse is not true, however: a matrix with repeated eigenvalues *may* still be diagonalizable.)

(d) **May**. Even if there is some $|\lambda_k| > 1$, you can still get decaying $A^n x$ if x is chosen to be an eigenvector x_j of a different eigenvalue with $|\lambda_j| < 1$, or to be a linear combination of such eigenvectors.

(e) **May**. For $e^{At}x$ to decay, all of its eigenvalues must have *negative real parts*. This is unrelated to the magnitude $|\lambda|$. For example, it could have an eigenvalue $\lambda = -2$.

(f) **Must**. $\begin{pmatrix} -3 \\ -6 \\ -9 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and all nonzero multiples of an eigenvector are also eigenvectors (of the same eigenvalue).

Problem 4 (25 points):

Suppose A is a real-symmetric matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 0$, and $\lambda_4 = 7$, with corresponding eigenvectors:

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \; x_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \; x_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Now, we construct a sequence of vectors $y_0, y_1, y_2, \ldots$ where each vector y_{k+1} in the sequence is computed from the previous vector y_k by solving

$$(A - 2I)y_{k+1} = y_k$$

for y_{k+1}. If $y_0 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, **give a good approximation** for y_{100}.

Solution:

Rearranging, we have $y_{k+1} = (A - 2I)^{-1} y_k$, so

$$y_k = (A - 2I)^{-k} y_0.$$

For $k = 100$, this will be dominated by the largest $|\lambda|$ eigenvalues of $(A - 2I)^{-1}$, but this matrix has the **same eigenvectors** as A with its eigenvalues λ replaced by $\frac{1}{\lambda - 2}$. So, the eigenvalues of $(A - 2I)^{-1}$ are

$$\frac{1}{\lambda_1 - 2} = -1, \; \frac{1}{\lambda_2 - 2} = 1, \; \frac{1}{\lambda_3 - 2} = -\frac{1}{2}, \text{ and } \frac{1}{\lambda_4 - 2} = \frac{1}{5}.$$

Of these, the largest magnitudes are -1 and $+1$, which both have magnitude 1, so y_{100} will be dominated by the x_1 and x_2 terms in the expansion of y_0. More explicitly, if we expand y_0 in the basis of eigenvectors:

$$y_0 = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \,,$$

then

$$y_{100} = (A - 2I)^{-100} y_0 = (-1)^{100} c_1 x_1 + 1^{100} c_2 x_2 + \left(-\frac{1}{2}\right)^{100} c_3 x_3 + \left(\frac{1}{5}\right)^{100} c_4 x_4 \approx c_1 x_1 + c_2 x_2 \,.$$

To compute this explicitly, we merely need to compute c_1 and c_2. But A is Hermitian and hence the eigenvectors must be (and are) **orthogonal**, so we just need **orthogonal projection** to compute the coefficients of the basis expansion:

$$c_1 = \frac{x_1^T}{x_1^T x_1} y_0 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \frac{5}{2},$$

$$c_2 = \frac{x_2^T}{x_2^T x_2} y_0 = \frac{1}{4} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2}.$$

Therefore,

$$y_{100} \approx \frac{5}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 2 \\ 3 \\ 2 \end{pmatrix}}.$$

Note that the next biggest term is on the order of $\frac{1}{2^{100}} \approx 7.9 \times 10^{-31}$, so this approximation is pretty darn good! Actually, the $c_3 = 1$ term is the only correction, since $c_4 = 0$ ($x_4^T y_0 = 0$).