MIT 18.06 Exam 2, Spring 2022
Johnson

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/26
2	/24
3	/25
4	/25
total	/100

Problem 1 (26 points):

You are given 5 data points $(x_k, y_k, z_k) \in \{(1,2,7),(0,0,2),(-1,0,3),(1,1,4),(2,-1,5)\}$. You want to find the least-squaare fit of these points to a plane:

$$f(x,y) = \alpha x + \beta y + \gamma$$

for some scalar parameters (α, β, γ). That is, you want to minimize $\sum_k \left[z_k - f(x_k, y_k)\right]^2$.

Write a linear equation whose solution is the unknown parameters (α, β, γ), of the form

$$(\text{some matrix}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = (\text{some vector})\,.$$

Give the matrix and the vector. **You can leave your matrix and/or vector as a product** of some other matrices—you don't need to multiply them out, and you don't need to solve the equation.

(blank page for your work if you need it)

Problem 2 (24 points):

Suppose that the columns of Q, given by

$$Q = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

are an orthonormal basis for $C(A)$ for some $A = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$.

(a) The vector $b = \begin{pmatrix} 5 \\ -6 \\ -6 \\ 1 \end{pmatrix}$ lies in $C(A)$. Write b in the basis of q_1, q_2, q_3.

(b) What is the orthogonal projection of $y = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix}$ onto $N(A^T)$?

(c) If Q was constructed by Gram–Schmidt applied to A, which of the following dot products *must* be zero? (Circle the zero terms.)

$$\begin{matrix} q_1^T a_1 & q_1^T a_2 & q_1^T a_3 \\ q_2^T a_1 & q_2^T a_2 & q_2^T a_3 \\ q_3^T a_1 & q_3^T a_2 & q_3^T a_3 \end{matrix}$$

(blank page for your work if you need it)

Problem 3 (25 points):

Suppose

$$f(x) = (b - Ax)^T M (b - Ax)$$

for some $m \times n$ matrix A, an $m \times m$ matrix $M = M^T$ (symmetric), and vectors $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

Write a linear equation satisfied by x when $\nabla f = 0$ (i.e. at extrema of f).

Problem 4 (25 points):

Answer the following questions, which should require **little or no computation.**

(a) If $A = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$, the projection matrix onto $C(A)$ is given by $\frac{a_1 a_1^T}{a_1^T a_1} + \frac{a_2 a_2^T}{a_2^T a_2}$ only when a_1 and a_2 are ____________________________________.

(b) If S and T are orthogonal subspaces of a vector space V, then

(i) their intersection (vectors in both S and T) is the set ____________

(ii) (dimension of S) + (dimension of T) must be (**circle one**):

$$= \text{ or } \leq \text{ or } \geq \text{(dimension of } V\text{)}$$

(c) For the vector space $\mathbb{R}^3$, give projection matrices onto:

(i) any 0-dimensional subspace

(ii) any 1-dimensional subspace

(iii) any 3-dimensional subspace

(d) Give an example Q matrix with orthonormal columns such that **either** $Q^T Q$ **or** QQ^T (circle **one**) is **not** equal to I.

(e) A is a 7×5 matrix of rank 4.

(i) Give the **size** _ × _ and **rank** of the following projection matrices:

i. P_1 = projection onto $C(A)$

ii. P_2 = projection onto $C(A^T)$

iii. P_3 = projection onto $N(A)$

iv. P_4 = projection onto $N(A^T)$

(ii) Give a sum or product of *two* of these P matrices that must $= 0$ (a zero matrix).

(iii) Give a sum or product of *two* of these P matrices that must $= I$ (an identity matrix).

(blank page for your work if you need it)